Linear Algebra Exercises

Problem 1

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \beta \\ -1 \end{bmatrix}$$

Find the parameters α and β so that the inhomogeneous linear system $A \cdot x = b$ has infinetely many solutions.

Hint. $\alpha = 0$ and $\beta = -1$

Problem 2

Find the rank of the above matrix A as a function of the parameter α.

Hint. $\operatorname{rank} A = 3$, if $\alpha \neq 0$ and $\operatorname{rank} A = 2$, if $\alpha = 0$

Problem 3

Find the diagonal form D of the symmetric matrix A below. Also determine the orthogonal matrix U so that $U^T A U = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Hint.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Problem 4

Find the tangent plane to the graph of the function f below at the point $P(2, 1, 4)$.

$$f(x, y) = 2y^2\sqrt{x^2 + y^2 + 4} - 2$$

Hint.
$f_1'(2, 1) = 4/3$
$f_2'(2, 1) = 38/3$
The equation of the tangent plane is: $4(x - 2) + 38(y - 1) - 3(z - 4) = 0$.

Problem 5

Specify the critical points of the function f, where:

$$f(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{xy}{8} - 2z^2$$

Hint. The only critical point is: $P(2, 2, 0)$, which is the single solution to

$$\begin{aligned} f_1'(x, y, z) &= -\frac{1}{x^2} + \frac{y}{8} = 0 \\ f_2'(x, y, z) &= -\frac{1}{y^2} + \frac{x}{8} = 0 \\ f_1'(x, y, z) &= -4z = 0 \end{aligned}$$

Problem 6

Find the Hesse-matrix at all critical points above. Classify them, if we have a minimum, maximum or saddle point.

Hint.

$$H = \begin{bmatrix} 1/4 & 1/8 & 0 \\ 1/8 & 1/4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

The eigenvalues of the Hessian are $\lambda_1 = 1/8$, $\lambda_2 = 3/8$ and $\lambda_3 = -4$. Consequently, H is indefinite, and the critical point is a *saddle point.*

Problem 7

Indicate whether the statement is TRUE or FALSE: **F T F F**

- If the linear system $A \cdot x = b$ admits a solution then the columns of A are linearly independent.
- For every square matrix $\det A) \cdot \det A^T) \geq 0$.
- If the vectors a_1, a_2, a_3 are linearly independent, and we set $b_1 = a_1 + a_2$, $b_2 = a_1 + a_2 + a_3$ and $b_3 = a_2 + a_3$, then the vectors b_1, b_2, b_3 are linearly independent as well.
- If x_0 is a strict local minimum point of the twice continuously differentiable function f, then the Hesse-matrix is positive definite at x_0.

Problem 8

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \beta \\ -1 \end{bmatrix}$$

Find all parameters α and β so that the linear system of equations $A \cdot x = b$ has a unique solution.

Hint. $\alpha \neq 0$ and β is any real number.

Problem 9

Consider the matrix A in the previous problem.

(a) Find all values of α so that $\det A \neq 0$.

(b) Find $\operatorname{rank} A$ and $\deg A$ as a function of the parameter α.

Hint. (a) $\alpha \neq 0$.

(b) If $\alpha \neq 0$, then $\operatorname{rank} A = 3$ and $\deg A = 0$. If $\alpha = 0$, then $\operatorname{rank} A = 2$ and $\deg A = 1$.

Problem 10

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ -1 & 1 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ \beta \end{bmatrix}$$

Find all parameters α and β so that the linear system $A \cdot x = b$ has at least one solution.

Hint. $\alpha = 0$ and $\beta = -1$.

Problem 11

Find the diagonal form D of the symmetric matrix A, and determine the orthogonal matrix U for which $U^T AU = D$.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Hint.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Problem 12

Let us assume that the pairs of vectors $a+b$ and $2a-2b$, moreover b and $6a+3b$ are orthogonal respectively. Find the angle of the vectors a and b, if none of them is zero vector.

Problem 13

The slope of the tangent line to the graph of $e^{-xy} + 2x + 3xy = 3$ at the point $(p, 0)$ is -1. What can we say about the value of the parameter p?

Problem 14

Evaluate the gradient vector of f at the point $(0, 1, 0)$, if

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\,\ln(ye^x)$$

Problem 15

Find the equation of the tangent plane to the graph of $f(x, y) = (4x^2 - 2y^3)\sqrt{x^2 + y^2 + 2}$ at the point $(1, 1, 4)$.

Problem 16

Consider the square matrix $A = [a_1, a_2, a_3]$. Find $\det A$ in terms of the determinant of the square matrix

$$B = [a_2 + a_3, a_1 + a_3, a_1 + a_2]$$

Problem 17

Locate all critical points of the function below:

$$f(x, y, z) = x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} + e^{z^2}$$

Hint.

$$\begin{aligned} f_1'(x, y, z) &= -\frac{1}{x^2} + \frac{y}{8} = 0 \\ f_2'(x, y, z) &= -\frac{1}{y^2} + \frac{x}{8} = 0 \\ f_1'(x, y, z) &= -4z = 0 \end{aligned}$$

There is a single critical point: $P(2, 2, 0)$.

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Find the Hesse-matrix of the function in the previous problem at every critical point, and decide which is a minimum, maximum or saddle point.

Hint.

$$H = \begin{bmatrix} 1/4 & 1/8 & 0 \\ 1/8 & 1/4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Problem 19

Set $f(x, y) = x - 3y$. Find the solution to the conditional extremum problem

$$\begin{aligned} & f(x,y) \to \max(\min) \\ & x^2 + y^2 = 9 \end{aligned}$$

Problem 20

Find the diagonal form D of the symmetric matrix A, and specify the orthogonal matrix U such that $U^T AU = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Hint.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Problem 21

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 4 \\ \beta \end{bmatrix}$$

Find all parameters α and β so that the linear system $A \cdot x = b$ has a unique solution.

Hint. $\alpha = 0$ and $\beta = -1$

Problem 22

Find the diagonal form D of the symmetric matrix A, and determine the orthogonal matrix U such that $U^T AU = D$.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

Hint.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Problem 23

Locate all critical points of the function:

$$f(x, y, z) = (2x^2 + y^2)e^{-z}$$

Hint. There is a single critical point: $P(2, 2, 0)$, which is the only solution to the system

$$\begin{aligned} f_1'(x,y,z) &= -\frac{1}{x^2} + \frac{y}{8} = 0 \\ f_2'(x,y,z) &= -\frac{1}{y^2} + \frac{x}{8} = 0 \\ f_1'(x,y,z) &= -4z = 0 \end{aligned}$$

Problem 24

Find the Hesse-matrix of the function in the previous problem at every critical point, and decide whether it is a minimum, maximum or saddle point.

Hint.

$$H = \begin{bmatrix} 1/4 & 1/8 & 0 \\ 1/8 & 1/4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Problem 25

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 2 & \alpha \\ 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}$$

Find all parameters α and β so that the linear system $A \cdot x = b$ has at least one solution.

Hint. $\alpha = 0$ and $\beta = -1$.

Problem 26

Find the diagonal form D of the symmetric matrix A and also find the othogonal matrix U for which $U^T AU = D$.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hint.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Problem 27

Find all critical points of the function:

$$f(x, y, z) = x \cdot e^x + \sqrt{1 + y^2 + z^2}$$

Hint.

$$\begin{aligned} f_1'(x, y, z) &= -\frac{1}{x^2} + \frac{y}{8} = 0 \\ f_2'(x, y, z) &= -\frac{1}{y^2} + \frac{x}{8} = 0 \\ f_1'(x, y, z) &= -4z = 0 \end{aligned}$$

The unique critical point is: $P(2, 2, 0)$

Problem 28

Find the Hesse-matrix of the function in the previous problem at every critical point, and decide whether it is a minimum, maximum or saddle point.

Hint.

$$H = \begin{bmatrix} 1/4 & 1/8 & 0 \\ 1/8 & 1/4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Problem 29

Examine if the vectors below are linearly in dependent, and find the value of

$$\dim \operatorname{lin}\{a_1, a_2, a_3, a_4\}$$

where

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} -1 \\ 8 \\ -9 \\ 4 \end{bmatrix} \quad a_4 = \begin{bmatrix} 5 \\ -5 \\ 10 \\ 1 \end{bmatrix}$$

Problem 30

Consider the following vectors:

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

and find out if the relation $b \in \operatorname{lin}\{a_1, a_2, a_3\}$ is true.

Problem 31

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & \alpha \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ \beta \end{bmatrix}$$

Find all values of the parameters α and β so that the inhomogeneous linear system $Ax = b$

(a) has infinitely many solutions,

(b) has no solution,

(c) has a unique solution.

Problem 32

Consider the linear transformation:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

and find the rank of A, and determine the product of all eigenvalues.