Calculus Problems and Exercises

Problem 1

Find out if the sequence below is monotone, bounded, and find its limit:

$$a_n = \frac{2n-1}{2n+3}$$

Hint. Rewrite the sequence in the following form:

$$a_n = \frac{2n+3-4}{2n+3} = 1 - \frac{4}{2n+3}.$$

Then obviously the sequence is monotone increasing, its upper bound is 1 (and this is the lowest possible), further the limit is 1:

$$\lim_{n\to\infty} a_n = 1$$

Problem 2

Determine the limit of the following sequence:

$$a_n = \frac{5n^2-3n+6}{3n^2+9n-4}$$

Hint. Factor out the highest degree of n (i.e. n^2) both from the numerator and the denominator, then we get

$$a_n = \frac{n^2}{n^2} \cdot \frac{5-3/n+6/n^2}{3+9/n-4/n^2}$$

Then the first factor is 1, while in the second factor the limit of the numerator is 5, and the limit of the denominator is 3. Therefore

$$\lim_{n\to\infty} a_n = \frac{5}{3}$$

Problem 3

Consider now a sequence with parameters, and calculate the limit:

$$a_n = \frac{\alpha_k n^k + \alpha_{k-1} n^{k-1} + \ldots + \alpha_0}{\beta_m n^m + \beta_{m-1} n^{m-1} + \ldots + \beta_0}$$

where the coefficients α_k and β_m are not zero.

Hint. Following the argument of the previous problem, factor out the highest degree of n in the numerator and in the denominator. Then we obtain

$$a_n = \frac{n^k}{n^m} \cdot \frac{\alpha_k + \alpha_{k-1}/n + \ldots + \alpha_0/n^k}{\beta_m + \beta_{m-1}/n + \ldots + \beta_0/n^m}$$

Observe that the limit of the second factor is α_k/β_m, since all the non-constant terms converge to zero. We can come up with the following statement:

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } k < m \\ \frac{\alpha_k}{\beta_m} & \text{if } k = m \\ +\infty & \text{if } k > m \text{ and } \alpha_k/\beta_m > 0 \\ -\infty & \text{if } k > m \text{ and } \alpha_k/\beta_m < 0 \end{cases}$$

Problem 4

Find the limit of the sequence below:

$$a_n = \frac{-5n^3 + 3n^2 - 8n + 6}{3n^2 - 12n + 21}$$

Hint. Based on the result in the previous problem, here we have $k > m$, and in addition the quotient of the coefficients in the highest degree terms is negative, thus:

$$\lim_{n\to\infty} a_n = -\infty$$

Problem 5

Examine whether the following sequence is monotone, bounded and find its limit:

$$a_n = \sqrt{n^2+2} - n$$

Hint. Each term tends to infinity, so carry out the following transformation::

$$a_n = (\sqrt{n^2+2} - n) \cdot \frac{\sqrt{n^2+2}+n}{\sqrt{n^2+2}+n} = \frac{n^2+2-n^2}{\sqrt{n^2+2}+n} = \frac{2}{\sqrt{n^2+2}+n} < \frac{2}{2n} = \frac{1}{n}$$

In view of the Squeezing Theorem, this sequence converges to zero. It is easy to see that the sequence is monotone decreasing, and its lower bound is zero (highest possible lower bound).

Problem 6

Create a sequence in the following recursive way: $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, and so forth

$$a_n = \sqrt{2+\sqrt{2+\dots\sqrt{2}}}$$

where in the formula for a_n there are n pieces of radicals. This is how a_n is created from a_{n-1} for $n = 2, 3, \dots$:

$$a_n = \sqrt{2 + a_{n-1}}$$

Find out if the sequence is monotone, bounded, and if so, find the limit!

Hint. It is easy to see that the sequence is strictly monotone increasing, since at every step we insert a new radical.

By using Mathematical Induction, we show that the sequence is bounded from above, and 2 is an upper bound. Indeed, in the first step $a_1 < 2$. Now if for any n we have $a_{n-1} < 2$, then for the n-th element we have

$$a_n^2 = 2 + a_{n-1} < 4$$

which implies that $a_n < 2$ for every index n.

It follows that the sequence is convergent, and let us denote by A the unknown (but positive) limit, i.e. $a_n \to A$. by the rule of creating the sequence we deduce for every $n \geq 2$

$$a_n^2 = 2 + a_{n-1}$$

where $a_n^2 \to A^2$, and $2 + a_{n-1} \to 2 + A$. Since the sides are equal for every n, we get

$$A^2 = 2 + A$$

The only positive solution of this quadratic equation is $A = 2$, therefore

$$\lim_{n\to\infty} a_n = 2$$

ATTENTION!

In the first two steps of the solution, proving the monotonicity and the boudedness of the sequence are profoundly important! Without showing this, the sequence is not necessarily convergent. For example, if we consider the sequence in which $a_1 = 1$, and

$$a_n = 2a_{n-1}$$

then by jumping to the equation for the unknown A (and skipping the first two steps) we have

$$A = 2A$$

whose only solution is $A = 0$, and that would mean that the limit is zero. However, this sequence is not bounded, its explicit form is

$$a_n = 2^{n-1}$$

and this is clearly not convergent, it tends to infinity.

Problem 7

Find the limit of the following sequence:

$$a_n = \left(\frac{2n+1}{2n+3}\right)^{n+1}$$

Hint. Inside the parentheses, divide both the numerator and the denominator of the fraction by $2n$:

$$a_n = \left(\frac{1+\frac{1/2}{n}}{1+\frac{3/2}{n}}\right)^{n+1} = \frac{\left(1+\frac{1/2}{n}\right)^n}{\left(1+\frac{3/2}{n}\right)^n} \cdot \frac{1+\frac{1/2}{n}}{1+\frac{3/2}{n}}$$

In the first factor, the limit of the numerator is $e^{1/2}$, while the limit of the denominator is $e^{3/2}$. The limit of the second factor is obviously 1, consequently

$$\lim_{n\to\infty} a_n = \frac{e^{1/2}}{e^{3/2}} = \frac{1}{e}$$

Problem 8

Determine the limit of the sequence below:

$$a_n = \left(1 - \frac{1}{n}\right)^{n^2}$$

Hint. we can rewrite the sequence in this form:

$$a_n = \left[\left(1 - \frac{1}{n}\right)^n\right]^n$$

The limit of the sequence inside the brackets is $1/e \approx 0.367891...$, and that means that starting from some index N we have

$$0.3 < \left(1 - \frac{1}{n}\right)^n < 0.4$$

for each $n \geq N$. By taking the n-th power of both sides

$$0.3^n < \left[\left(1 - \frac{1}{n}\right)^n\right]^n < 0.4^n$$

As we see, both the lower estimate and the upper estimate tend to zero. Therefore, by the Squeezing Theorem we conclude

$$\lim_{n\to\infty} a_n = 0$$

Problem 9

Find the limit of the sequence $a_n = \sqrt[n]{n}$.

Hint. Attention! This sequence is not monotone, verify it by direct calculation!

As a matter of fact, every element is geater than 1, so it is legal to write them this way:

$$a_n = \sqrt[n]{n} = 1 + h_n$$

where $h_n > 0$ for every n. Raise both sides to the power of n:

$$n = (1 + h_n)^n > 1 + \binom{n}{2} h_n^2 = 1 + \frac{n(n-1)}{2} h_n^2$$

where on the right-hand side we kept only the first and third terms from the Binomial Expansion, and discarded all other positive terms. Regrouping the inequality we obtain

$$0 < h_n^2 < \frac{2}{n}$$

By making use of the Squeezing Theorem, we conclude that $h_n^2 \to 0$, and hence $h_n \to 0$. Thus

$$\lim_{n\to\infty} a_n = 1$$

ATTENTION!

Please make sure that keeping only the first and second terms from the Binomial Expansion is not sufficient, we definitely need the quadratic term as well.

Problem 10

Compute the limit of the sequence below:

$$a_n = \frac{4^{n+1} - 2 \cdot 6^n + 5 \cdot 3^n}{3 \cdot 6^n - 7 \cdot 5^{n+1} + 2^n}$$

Hint. Factor out the power with the highest base (i.e. 6^n) from the numerator and the denominator. What we get is:

$$a_n = \frac{4 \cdot (4/6)^n - 2 + 5 \cdot (3/6)^n}{3 - 35 \cdot (5/6)^n + (2/6)^n}$$

Here the limit of the numerator os -2, since all the other terms tend to zero. Analogously, the limit of the denominator is 3, because the other two non-constant terms converge to zero. Thus:

$$\lim_{n\to\infty} a_n = -\frac{2}{3}$$

Problem 11

Find the sum of the infinite series below:

$$S = \sum_{k=1}^{\infty} \frac{2^{k+2}}{3^k}$$

Hint. If we factor out 4 from the sum, then get the terms of a geometric series with ratio 2/3. The sum of that series is:

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1 - 2/3} = 3$$

Be careful, in our problem the summation starts from $k = 1$ and the term that belongs to $k = 0$ is missing. Thus:

$$S = \sum_{k=1}^{\infty} \frac{2^{k+2}}{3^k} = 4(3-1) = 8$$

Problem 12

Examine the convergence of the following series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$$

Hint. Divide the expression behind the sum sign into the difference of two fractions:

$$\frac{1}{k^2+2k} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$

Then the n-th partial sum can be written like:

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+2k} = \frac{1}{2} \cdot \left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+2}\right)\right]$$

Observe that the negative and positive terms (in every second parenthesis) cancel each other. Only the first two positive and the last two negative terms stay alive:

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+2k} = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

This sequence is obviously convergent and its limit is $3/4$. It follows that the series is convergent, and its sum is

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2+2k} = \frac{3}{4}$$

Problem 13

Find the sum of the following series:

$$\sum_{k=2}^{\infty} \frac{1}{k^3-k}$$

Hint. As in the previous problem, we would like to express the term behind the sum sign as the difference of fractions, but this is not so simple in this case. We do this:

$$\frac{1}{k^3-k} = \frac{A}{k-1} + \frac{B}{k} + \frac{C}{k+1}$$

where A, B and C are unknown constants. Find the common denominator on the right-hand side:

$$\frac{1}{k^3-k} = \frac{A(k^2+k)+B(k^2-1)+C(k^2-k)}{k^3-k} = \frac{(A+B+C)k^2+(A-C)k-B}{k^3-k}$$

Since the sides are identically equal, we get the system of equations:

$$\begin{aligned} A+B+C &= 0 \\ A-C &= 0 \\ -B &= 1 \end{aligned}$$

The solutions are $A = C = 1/2$ and $B = -1$. So, the n-th partial sum is:

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \frac{1}{k^3-k} = \frac{1}{2} \cdot \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{2}{k} + \frac{1}{k+1}\right) \\ &= \frac{1}{2} \cdot \left[\left(1-1+\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)\right] \end{aligned}$$

As we can see, every negative term is cancelled by the last positive term of the preceding parentheses and the first positive term of the succeeding parentheses. The terms that remain are:

$$S_n = \frac{1}{2} \cdot \left(1-1+\frac{1}{2}+\frac{1}{n}-\frac{2}{n}+\frac{1}{n+1}\right)$$

This shows us that this sequence is convergent, and its limit is $1/4$, hence

$$\sum_{k=2}^{\infty} \frac{1}{k^3-k} = \frac{1}{4}$$

Problem 14

Is the series below convergent?

$$\sum_{k=1}^{\infty} \frac{k+2}{k^2+k+2}$$

Hint. It is easy to verify that

$$\frac{k+2}{k^2+k+2} > \frac{1}{k+1}$$

for each $k \in \mathbb{N}$. The fractions on the right-hand side are the terms of the Harmonic series, with the exclusion of the very first term. The Harmonic series is divergent, so, based on our sufficient condition, our example is divergent as well.

ATTENTION!

It is worth noting that in the case of series of this type, if the magnitude of the fraction is k^{-1}, then the series is divergent, if however, the magnitude is $k^{-\alpha}$, where $\alpha > 1$, then the series is convergent. Summing up, it is vitally important how fast the k-th term converges to zero. We revisit these questions in Chapter 9.

Problem 15

As we have seen in the lecture,

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

is convergent, and for the sum we have $S < 2$. To prove this for every $k \geq 2$ we applied the upper estimate

$$\frac{1}{k^2} < \frac{1}{k^2-k}$$

Could we perhaps find a better estimate for the unknown sum?

Hint. Use the following more accurate estimate for the terms: $k \geq 2$

$$\frac{1}{k^2} < \frac{1}{k^2-1}$$

for every $k \geq 2$. This is how we divide the right-hand side into the difference of two fractions:

$$\frac{1}{k^2-1} = \frac{1}{2} \cdot \left(\frac{1}{k-1} - \frac{1}{k+1} \right)$$

Then, for the n-th partial sum we obtain:

$$S_n = \frac{1}{2} \cdot \left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \right]$$

Each negative term will be cancelled by the positive term in the second parentheses. What remains is:

$$S_n = \frac{1}{2} \cdot \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

which is convergent, and its limit is $3/4$. Therefore, the n-th partial sum of the series in the problem:

$$\sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{k^2-1} < 1 + \frac{3}{4} = \frac{7}{4}$$

for every $n \in \mathbb{N}$. Consequently

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2} < \frac{7}{4}$$

Remark: the exact value of the sum is $S = \pi^2/6$ (but this is hard to prove!).

Problem 16

Decide whether or not the following series is convergent:

$$\sum_{k=1}^{\infty} \frac{k^3}{2^k}$$

Hint. Use the Quotient-test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^3}{2^{k+1}} \cdot \frac{2^k}{k^3} = \left(\frac{k+1}{k}\right)^3 \cdot \frac{1}{2}$$

If $k \to \infty$ then the first cubic term converges to 1, thus:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{2} < 1$$

We conclude that the series is convergent (but we do not know what the sum is).

Problem 17

An arbitrary real number $x \neq 0$ is given, and examine the convergence of the series:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Hint. Exploit again the Quotient-test (keep in mind that x can be negative, so use absolute values):

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \frac{|x|}{k+1}$$

If $k \to \infty$ then the expression on the right-hand side tends to zero, which is less than 1. Therefore, the series is absolutely convergent for each real number x, and henceforth convergent as well.

Problem 18

Find out if the series below is convergent:

$$\sum_{k=1}^{\infty} \left(\frac{k-1}{k+3}\right)^2$$

Hint. Try to use the Quotient-test again:

$$\frac{a_{k+1}}{a_k} = \left(\frac{k}{k+4}\right)^2 \cdot \left(\frac{k+3}{k-1}\right)^2 = \left(\frac{k^2+3k}{k^2+3k-4}\right)^2$$

The last fraction converges to 1 if $k \to \infty$. Indeed, the quotient of the coefficients of the quadratic terms is 1. The Quotient-test gives us 1 as a limit, so, based purely on this, nothing can be said about the convergencs.

However, if we take a look at the terms of the series, we can see that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(\frac{k-1}{k+3}\right)^2 = 1$$

which means that a_k does not converge to zero. The necessary condition is not satisfied, so the series is divergent.

Problem 19

Examine the convergence of the series:

$$\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k$$

Hint. Observe that for the base of the power we have:

$$0 \leq \frac{k-1}{2k+3} < \frac{1}{2}$$

By taking the k-th power of both sides we get:

$$0 \leq \left(\frac{k-1}{2k+3}\right)^k < \left(\frac{1}{2}\right)^k$$

for all $k \in \mathbb{N}$. On the right-hand side of the inequality we have the terms of the geometric series with ratio $1/2$, which is convergent, and its sum is

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-1/2} - 1 = 1$$

Here in the sum formula for geometric series we need to subtract 1, because the term $k = 0$ is missing. Making use of the sufficient condition we deduce that the series is convergent, and for the sum we have:

$$\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k < 1$$

ATTENTION!
As a practice, solve the problem by applying the Quotient-test! We will come up with the same answer. Indeed:

$$\frac{a_{k+1}}{a_k} = \left(\frac{k}{2k+5}\right)^{k+1} \cdot \left(\frac{2k+3}{k-1}\right)^k = \left(\frac{2k+3}{2k+5}\right)^k \cdot \left(\frac{k}{k-1}\right)^k \cdot \frac{k}{2k+5}$$

If $k \to \infty$ then the first factor tends to e^{-1}, the second to e, and the third to $1/2$. Therefore the limit is less than 1, and this implies that the series is convergent.

ATTENTION!
The advantage of the first solution is that it provides an estimate for the sum, while the second does not!

Problem 20

Is the series below convergent?

$$\sum_{k=1}^{\infty} \left(\frac{k-1}{k+1}\right)^k$$

Hint. Try again using the Quotient-test:

$$\frac{a_{k+1}}{a_k} = \left(\frac{k}{k+2}\right)^{k+1} \cdot \left(\frac{k+1}{k-1}\right)^k = \left(\frac{k+1}{k+2}\right)^k \cdot \left(\frac{k}{k-1}\right)^k \cdot \frac{k}{k+2}$$

If $k \to \infty$, then the first factor converges to e^{-1}, the second to e, and the third to 1. Hence, the limit is 1, and this way we cannot say anything about the convergence.
Examine however the terms of the series:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(\frac{k-1}{k+1}\right)^k = \frac{1}{e^2}$$

that is a_k does not converge to zero. The necessary condition is violated, so this series is divergent.

Problem 21

Find the following limit:

$$\lim_{x\to+\infty} \frac{-3x^3 + 2x^2 + 15x - 6}{2x^3 - 7x^2 + 18x - 8}$$

Hint. Factor out the highest degree of x from both the numerator and the denominator:

$$\lim_{x\to+\infty} \frac{x^3}{x^3} \cdot \frac{-3 + 2/x + 15/x^2 - 6/x^3}{2 - 7/x + 18/x^2 - 8/x^3}$$

The first factor is 1, while in the second factor every term tends to zero, except the constant terms. Hence, the limit of the fraction is $-3/2$. Therefore:

$$\lim_{x\to+\infty} \frac{-3x^3 + 2x^2 + 15x - 6}{2x^3 - 7x^2 + 18x - 8} = -\frac{3}{2}$$

Problem 22

The following problem is given with parameters. Find the limit:

$$\lim_{x\to+\infty} \frac{\alpha_k x^k + \alpha_{k-1}x^{k-1} + \ldots + \alpha_0}{\beta_m x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_0}$$

Hint. Just like in the preceding problem, factor out the highest degree from the numerator and the denominator:

$$\lim_{x\to+\infty} \frac{x^k}{x^m} \cdot \frac{\alpha_k + \alpha_{k-1}/x + \ldots + \alpha_0/x^k}{\beta_m + \beta_{m-1}/x + \ldots + \beta_0/x^m}$$

The limit of the second fraction is α_k/β_m, because both in the numerator and the denominator every non-constant term tends to zero. Thus:

$$\lim_{x\to+\infty} \frac{\alpha_k x^k + \alpha_{k-1}x^{k-1} + \ldots + \alpha_0}{\beta_m x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_0} = \begin{cases} 0 & \text{if } k < m \\ \frac{\alpha_k}{\beta_m} & \text{if } k = m \\ +\infty & \text{if } k > m \text{ and } \frac{\alpha_k}{\beta_m} > 0 \\ -\infty & \text{if } k > m \text{ and } \frac{\alpha_k}{\beta_m} < 0 \end{cases}$$

Problem 23

In a completely analogous way, establish a formula for the limit below:

$$\lim_{x\to-\infty} \frac{\alpha_k x^k + \alpha_{k-1}x^{k-1} + \ldots + \alpha_0}{\beta_m x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_0}$$

Hint. Follow the steps of the solution in the previous problem, then we get:

$$\lim_{x\to-\infty} x^{k-m} \cdot \frac{\alpha_k + \alpha_{k-1}/x + \ldots + \alpha_0/x^k}{\beta_m + \beta_{m-1}/x + \ldots + \beta_0/x^m}$$

Keep in mind that an even power of x tends to $+\infty$ in $-\infty$, while an odd power of x tends to $-\infty$ in $-\infty$. Based on this observation, we have:

$$\lim_{x\to-\infty} \frac{\alpha_k x^k + \alpha_{k-1}x^{k-1} + \ldots + \alpha_0}{\beta_m x^m + \beta_{m-1}x^{m-1} + \ldots + \beta_0} = \begin{cases} 0 & \text{if } k < m \\ \frac{\alpha_k}{\beta_m} & \text{if } k = m \\ +\infty & \text{if } k > m \text{ and } k-m \text{ even, and } \frac{\alpha_k}{\beta_m} > 0 \\ -\infty & \text{ha } k > m \text{ és } k-m \text{ even, and } \frac{\alpha_k}{\beta_m} < 0 \\ -\infty & \text{if } k > m \text{ and } k-m \text{ odd, and } \frac{\alpha_k}{\beta_m} > 0 \\ +\infty & \text{if } k > m \text{ and } k-m \text{ odd, and } \frac{\alpha_k}{\beta_m} < 0 \end{cases}$$

Problem 24

Based on our previous result, find the limit below, just by "taking a close look":

$$\lim_{x\to-\infty} \frac{4x^5 - 6x^4 + x^3 - 13x^2 + 28x - 5}{-21x^2 + 14x - 22}$$

Hint. In this problem $k - m = 5 - 2 = 3 > 0$ is odd, further the quotient of the main coefficients is $-4/21$, which is negative, so apply the formula from the previous problem:

$$\lim_{x\to-\infty} \frac{4x^5 - 6x^4 + x^3 - 13x^2 + 28x - 5}{-21x^2 + 14x - 22} = +\infty$$

Problem 25

Determine the following limit:

$$\lim_{x\to 0} \frac{\sin 2x}{3x}$$

Hint. Rewrite the expression in this form:

$$\frac{\sin 2x}{3x} = \frac{\sin 2x}{2x} \cdot \frac{2}{3}$$

where $x \neq 0$. Then we immediately see:

$$\lim_{x\to 0} \frac{\sin 2x}{3x} = \frac{2}{3}$$

With different numbers the solution can be carried out in an analogous way.

Problem 26

Examine if the following limit exists:

$$\lim_{x\to 0} \frac{\sqrt{1-\cos^2 x}}{x}$$

Hint. If we take the square root, we obtain:

$$\frac{\sqrt{1-\cos^2 x}}{x} = \frac{|\sin x|}{x} = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0 \\ -\frac{\sin x}{x} & \text{if } x < 0 \end{cases}$$

since $\sin x$ is an odd function. Then the right-hand side limit is 1, while the left-hand side limit is -1. Then the two one-sided limits do not coincide, therefore, the limit does not exist.

Problem 27

Consider the function

$$f(x) = \frac{x+1}{x-2}$$

and determine the one-sided limits at $x = 2$.

Hint. Take a sequence $x_n \to 2$, $x_n < 2$ arbitrariliy. In that case the denominator is negative and tends to zero, while the numerator is positive and tends to 3. Thus, the left-hand limit is $-\infty$. We can argue very similarly to demonstrate that the right-hand limit is $+\infty$. Summing up:

$$\lim_{x\to 2-} \frac{x+1}{x-2} = -\infty \quad \text{and} \quad \lim_{x\to 2+} \frac{x+1}{x-2} = +\infty$$

Problem 28

Investigate the limit:

$$\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x$$

If $x > 0$ is a given number, then choose an integer n so that $n \leq x < n+1$. It can be shown that

$$\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n+1}\right)^{n+1}$$

Based on the facts that we studied about sequences, and by using the Squeezing Theorem, we can state:

$$\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$$

Using a very similar argument we can show that

$$\lim_{x\to -\infty} \left(1+\frac{1}{x}\right)^x = e$$

Moreover, if α is an arbitrarily given real number, then

$$\lim_{x\to \pm\infty} \left(1+\frac{\alpha}{x}\right)^x = e^\alpha$$

Making use of these observations find the following limit:

$$\lim_{x\to +\infty} \left(\frac{3x+5}{3x+2}\right)^x$$

Hint. Just like in the case of sequences, carry out the following transformation:

$$\left(\frac{3x+5}{3x+2}\right)^x = \frac{\left(1+\frac{5/3}{x}\right)^x}{\left(1+\frac{2/3}{x}\right)^x}$$

where we divided both the numerator and the denominator by $3x$. By using the results above, we conclude that the limit of the numerator is $e^{5/3}$, and the limit of the denominator is $e^{2/3}$. By taking the quotient of the two limits, we receive:

$$\lim_{x\to+\infty}\left(\frac{3x+5}{3x+2}\right)^x = \frac{e^{5/3}}{e^{2/3}} = e$$

Problem 29

Find all parameters a and b so that the function f is continuous everywhere:

$$f(x) = \begin{cases} \frac{b\sin x}{2x} & \text{if } x<0 \\ b-a & \text{if } x=0 \\ \frac{1-\cos x}{x^2}+b & \text{if } x>0 \end{cases}$$

Hint. It is immediately clear that f is continuous at all points $x \neq 0$. For the continuity at $x = 0$ we need the existence of the limit at this point, and limit must coincide with $f(0)$, the value of the function at $x = 0$.

The function has one-sided limits at $x = 0$. In particular, the left-hand limit is:

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\frac{b\sin x}{2x} = \frac{b}{2}$$

and the right-hand limit is:

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\frac{1-\cos x}{x^2} + b = b + \frac{1}{2}$$

The limit exists if and only if the one-sided limits agree, that is:

$$\frac{b}{2} = b + \frac{1}{2}$$

where the only solution is $b = -1$. Then the limit of f at $x = 0$ is

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} f(x) = -\frac{1}{2}$$

ATTENTION!

The existence of the limit at $x = 0$ has nothing to do with $f(0)$. The function does not even need to be defined at $X = 0$!

The function f is continuous at $x = 0$ if and only if its limit here is the same as its value, i.e.

$$f(0) = b - a = -1 - a = \lim_{x\to 0} f(x) = -\frac{1}{2}$$

from where we get $a = -1/2$.

Problem 30

Find the following limit:

$$\lim_{x\to\infty}(\sqrt{x^2+2}-x)$$

Hint. Both terms of the difference tend to infinity, so carry out the following transformation:

$$(\sqrt{x^2+2}-x)\cdot\frac{\sqrt{x^2+2}+x}{\sqrt{x^2+2}+x} = \frac{x^2+2-x^2}{\sqrt{x^2+2}+x} = \frac{2}{\sqrt{x^2+2}+x} < \frac{2}{2x} = \frac{1}{x}$$

By using the Squeezing Theorem, we immediately see that limit is zero:

$$\lim_{x\to\infty}(\sqrt{x^2+2}-x) = 0$$

(This problem is completely analogous to a problem on sequences. Scroll up and check it!)

Problem 31

Find the derivative of the following function:

$$F(x) = \left(2x^3 - 6x^2 + 12x - 8\right)^8$$

Hint. Introduce the notations:

$$f(y) = y^8 \quad \text{és} \quad g(x) = 2x^3 - 6x^2 + 12x - 8$$

then the function F can be given in the form $F = f \circ g$. Hence, by the Chain-Rule:

$$F'(x) = f'(g(x)) \cdot g'(x) = 8\left(2x^3 - 6x^2 + 12x - 8\right)^7 \cdot \left(6x^2 - 12x + 12\right)$$

Problem 32

Calculate the derivative of the function below:

$$F(x) = \left(\frac{2x^2 - 5x + 6}{3x + 7}\right)^6$$

Hint. Now, if we use the following notations:

$$f(y) = y^6 \quad \text{és} \quad g(x) = \frac{2x^2 - 5x + 6}{3x + 7}$$

then we can see that $F = f \circ g$. Making use of the Chain-Rule and the Quotient-Rule we get:

$$F'(x) = 6\left(\frac{2x^2 - 5x + 6}{3x + 7}\right)^5 \cdot \frac{(4x - 5)(3x + 7) - (2x^2 - 5x + 6) \cdot 3}{(3x + 7)^2}$$

This latter expression can further be simplified.

Problem 33

Find the derivative of the following product:

$$F(x) = (3x - 7)^8 \cdot (5x + 1)^6$$

Hint. We differentiate the powers as we did in the previous exercises. Further, we exploit the Product-Rule:

$$F'(x) = 8(3x - 7)^7 \cdot 3 \cdot (5x + 1)^6 + (3x - 7)^8 \cdot 6(5x + 1)^5 \cdot 5$$

Problem 34

Determine the derivative of the function $f(x) = \sin x$ at an arbitrary point $x \in \mathbb{R}$.

Hint. As we have seen in today's lecture, f is differentiable at $x = 0$ and $f'(0) = 1$. Consider now the difference quotient of f at an arbitrary point x:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \end{aligned}$$

where we relied on the additive rule. In the first term of the second line

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} h \frac{\cos h - 1}{h^2} = 0 \cdot (-\frac{1}{2}) = 0$$

while in the second term

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

Therefore, the limit of the difference quotient is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0 \cdot \sin x + 1 \cdot \cos x$$

and that means $f'(x) = \cos x$.

Problem 35

Find the derivative function of $f(x) = \cos x$.

Hint. In view of the identity $f(x) = \cos x = \sin(x + \pi/2)$, the Chain-Rule yields:

$$f'(x) = \cos(x + \frac{\pi}{2}) \cdot 1 = -\sin x$$

Problem 36

Calculate the derivative of the tangent function, $f(x) = \tan x$.

Hint. As is well known,

$$f(x) = \tan x = \frac{\sin x}{\cos x} \quad \text{where} \quad x \neq \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

Apply the Quotient-Rule at all points of the domain:

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Problem 37

Consider the function

$$f(x) = (1 + \sin^2 x)(1 + \cos^2 x)$$

and find its derivative.

Hint. Apply the Product-Rule, and use the Chain-Rule for the quadratic trigonometric expressions. Then we obtain

$$f'(x) = 2 \sin x \cos x(1 + \cos^2 x) - (1 + \sin^2 x) \cdot 2 \cdot \cos x \sin x$$

By using the double angle formula, this expression can be given in a simplified way:

$$f'(x) = \sin 2x \cdot (\cos^2 x - \sin^2 x) = \sin 2x \cdot \cos 2x = \frac{1}{2} \sin 4x$$

ATTENTION!

When calculating derivatives of trigonometric functions we may get various results depending on the ways we found the derivative. If our calculations are correct, then obviously all results are identical (but sometimes this is not easy to recognize).

For instance, in the example above, first carry out the indicated multiplication, and then differentiate the product. Verify, that this way you have the same result.

Problem 38

Examine if the following function is differentiable at the point $x = 0$:

$$f(x) = |x| \cdot \sin x$$

Hint. Write the difference quotient at $x = 0$:

$$\frac{f(0 + h) - f(0)}{h} = \frac{|h| \cdot \sin h}{h} = \begin{cases} \sin h & \text{if } h > 0 \\ -\sin h & \text{if } h < 0 \end{cases}$$

It is easy to see that here the right-hand limit and the left-hand limit coincide, their common value is 0, and consequently, the difference quotient admits a limit at 0. Thus, the function is differentiable at $x = 0$ and the derivative is $f'(0) = 0$.

ATTENTION!

We cannot use the Product-Rule for proving the differentiability of f, because the first factor is not differentiable at the point $x = 0$!

Problem 39

Find the unknown parameter a so that the tangent line to the graph of the function

$$f(x) = 2x + ax^3$$

taken at $x = 1$ passes through the point $P(3, 20)$.

Hint. At $x = 1$ we have $f(1) = 2 + a$. On the other hand, the derivative of the function is

$$f'(x) = 2 + 3ax^2$$

and its value at $x = 1$ is $f'(1) = 2 + 3a$, this is the slope of the tangent line. The equation of the tangent line at $x = 1$ is given by

$$y = (2 + 3a)(x - 1) + 2 + a$$

The tangent line passes through the point $P(3, 20)$ if and only if the coordinates of P fulfill the equation of the tangent line. By substituting the coordinates we get the simple linear equation:

$$20 = (2 + 3a)(3 - 1) + 2 + a = 6 + 7a$$

where the only solution is $a = 2$.

Problem 40

Consider the cubic function:

$$f(x) = \frac{x^3}{6} - \frac{5x^2}{4} + \frac{3x}{2} + \frac{1}{12}$$

Find the point on the graph of the function, where the tangent line is perpendicular to the line with equation $y = 2x + 9$.

Hint. Let us denote the unknown point on the graph by $P(a, f(a))$. The slope of the tangent line at this point is

$$f'(a) = \frac{a^2}{2} - \frac{5a}{2} + \frac{3}{2}$$

The necessary and sufficient condition for perpendicularity of two straight lines is that their slopes are negative reciprocals of each other. That gives us

$$\frac{a^2}{2} - \frac{5a}{2} + \frac{3}{2} = -\frac{1}{2}$$

In a simplified for we get the quadratic equation

$$a^2 - 5a + 4 = 0$$

that possesses two solutions: $a_1 = 1$ and $a_2 = 4$. This tells us that we found two points with the given property: $P_1(1, 1/2)$, and $P_2(4, -41/12)$.

Problem 41

Find the derivative of the function below:

$$F(x) = \ln \sqrt{1 + x^2}$$

Hint. Verify that the function is defined on the whole real line (and it is nonnegative). Observe that F is the composition of three functions: the $1 + x^2$, the square root, and the logarithm, in this order. By applying the Chain-Rule:

$$F'(x) = \frac{1}{\sqrt{1 + x^2}} \cdot \frac{1}{2\sqrt{1 + x^2}} \cdot 2x = \frac{x}{1 + x^2}$$

We could have gotten the derivative easier, by using this identity:

$$F(x) = \ln \sqrt{1 + x^2} = \frac{1}{2} \ln(1 + x^2)$$

Please verify that this way we get the same result.

Problem 42

Calculate the derivative of the following function:

$$F(x) = (x^2 + 3x)e^{2x - x^2}$$

Hint. Use the product rule of differentiation, and keep in mind that we need the Chain-Rule when finding the derivative of the secod factor:

$$F'(x) = (2x+3)e^{2x-x^2} + (x^2+3x)(2-2x)e^{2x-x^2} = (-2x^3-4x^2+8x+3)e^{2x-x^2}$$

Problem 43

How can we find the derivative of this function:

$$F(x) = x^x \quad x > 0$$

Hint. Make sure that neither the rule for power functions, nor the rule for exponential functions can directly be applied. Therefore, we use this trick:

$$F(x) = x^x = \left(e^{\ln x}\right)^x = e^{x \ln x} \quad x > 0$$

In this form the Chain-Rule can be applied:

$$F'(x) = e^{x \ln x} \cdot (\ln x + 1) = x^x(\ln x + 1)$$

where we used the product rule when differentiating the exponent. We will follow the same method when we compute the derivatives of power function, where both the base and the exponent are functions of x.

Problem 44

Following the idea of the previous problem, find the derivative of this function:

$$F(x) = \sqrt[x]{x} \quad x > 0$$

Hint. Just like in the preceding exercise, rewrite the function in this form:

$$F(x) = \left(e^{\ln x}\right)^{1/x} = e^{\frac{1}{x}\ln x} \quad x > 0$$

Then carrying out a similar calculation, we get:

$$F'(x) = e^{\frac{1}{x}\ln x} \cdot \left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right) = \sqrt[x]{x} \cdot \frac{1}{x^2} \cdot (1 - \ln x)$$

Problem 45

Examine if the function

$$f(x) = xe^{-|x|}$$

is differentiable at $x = 0$.

Hint. Take the difference quotient at $x = 0$:

$$\frac{f(h) - f(0)}{h} = \begin{cases} e^{-h} & \text{ha } h > 0 \\ e^{h} & \text{ha } h < 0 \end{cases}$$

The one-sided limits of this expression exist, and both the right-hand and left-hand limits are 1. Therefore, the function is differentiable at 0, and $f'(0) = 1$.

Based on this observation, the function is differentiable everywhere, and its derivative is

$$f'(x) = \begin{cases} (1-x)e^{-x} & \text{ha } x \geq 0 \\ (1+x)e^{x} & \text{ha } x < 0 \end{cases}$$

ATTENTION! This very last formula can only be posted, if we verifyed the differentiability at $x = 0$! We cannot use the Chain-Rule for proving the differentiability of f, because the exponent is not differentiable at the point $x = 0$!

Problem 46

The function $f(x) = \sin x$ is not one-to-one, and hence, it does not have an inverse. However, if we restrict the domain to the interval $[-\pi/2, \pi/2]$, then the function becomes one-to-one (and its range is the interval $[-1, 1]$), and hence, it posseses an inverse. This inverse function is called the "arc sine" function, and its notation is

$$f^{-1}(y) = \arcsin y$$

ATTENTION! Sketch the graph of the inverse!

Let y be an interior point of the interval $[-1, 1]$, and find the derivative of the inverse function at y. Take the point $x \in [-\pi/2, \pi/2]$ for which $y = \sin x$. Since y was an interior point, therefore, x is going to be an interior point of the interval $[-\pi/2, \pi/2]$. For such interior points we have $f'(x) = \cos x \neq 0$. Thus, we can use the differentiability theorem for the inverse function:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

because in the entire open interval $(-\pi/2, \pi/2)$ we have $\cos x > 0$. Thus, we obtain

$$\left(f^{-1}\right)'(y) = \frac{1}{\sqrt{1-y^2}}$$

where $-1 < y < 1$.

Problem 47

The function $f(x) = \cos x$ is not one-to-one, and hence, it does not have an inverse. However, if we restrict the domain to the interval $[0, \pi]$, then the function becomes one-to-one (and its range is the interval $[-1, 1]$), and hence, it posseses an inverse. This inverse function is called the "arc cosine" function, and its notation is

$$f^{-1}(y) = \arccos y$$

ATTENTION! Sketch the graph of the inverse!

Let y be an interior point of the interval $[-1, 1]$, and find the derivative of the inverse function at y. Take the point $x \in [0, \pi]$ for which $y = \cos x$. Since y was an interior point, therefore, x is going to be an interior point of the interval $[0, \pi]$. For such interior points we have $f'(x) = -\sin x \neq 0$. Thus, we can use the differentiability theorem for the inverse function:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = -\frac{1}{\sin x} = -\frac{1}{\sqrt{1-\cos^2 x}} = -\frac{1}{\sqrt{1-y^2}}$$

because in the entire open interval $(0, \pi)$ we have $\sin x > 0$. Thus, we obtain

$$\left(f^{-1}\right)'(y) = -\frac{1}{\sqrt{1-y^2}}$$

where $-1 < y < 1$.

ATTENTION! It is interesting to notice that the derivatives of the functions $\arcsin x$ and $\arccos x$ are the negatives of each other, so the derivative of their sum is zero.

This is not surprising in view of the fact that for every x we have

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

This means that if for a given point $y \in (-1, 1)$ the point $x \in (0, \pi)$ fulfills

$$y = \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

then for the inverse functions we obtain

$$x = \arccos y \quad \text{moreover} \quad \frac{\pi}{2} - x = \arcsin y$$

Adding up the two equalities we get

$$\arccos y + \arcsin y = \frac{\pi}{2}$$

This tells us that the sum of the two functions is constant. Consequently, its derivative is zero.

Problem 48

Now we consider the tangent function $f(x) = \tan x$. This one is not invertible either, for it is not one-to-one. However, if the domain is restricted to the open interval $(-\pi/2, \pi/2)$, then we get a one-to-one function (whose range is the entire real line $(-\infty, \infty)$). The inverse of this function is called the "arc tangent", and its notation is:

$$f^{-1}(y) = \arctan y$$

whose domain is the whole real line $(-\infty, \infty)$.

ATTENTION! Please sketch the graph!

Recall that the derivative of the tangent function is:

$$f'(x) = \frac{1}{\cos^2 x} \neq 0$$

on the open interval $(-\pi/2, \pi/2)$, so we can apply the inverse differentiability theorem:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

for every point $y \in (-\infty, \infty)$.

Problem 49

Find the following limit:

$$\lim_{x\to 0} \frac{\sqrt{4+x^2}-2}{1-\cos x}$$

Hint. At $x = 0$ the fraction is of the form $0/0$, so use the L'Hôpital-Rule:

$$\lim_{x\to 0} \frac{\sqrt{4+x^2}-2}{1-\cos x} = \lim_{x\to 0} \frac{x/\sqrt{4+x^2}}{\sin x}$$

which is still of the form $0/0$. Apply the L'Hôpital-Rule again:

$$\lim_{x\to 0} \frac{x/\sqrt{4+x^2}}{\sin x} = \lim_{x\to 0} \frac{\frac{\sqrt{4+x^2}-x^2/\sqrt{4+x^2}}{4+x^2}}{\cos x}$$

In this expression the limit of the numerator is $1/2$, while the limit of the denominator is 1, hence

$$\lim_{x\to 0} \frac{\sqrt{4+x^2}-2}{1-\cos x} = \frac{1}{2}$$

Problem 50

By making use of the L'Hôpital-Rule, determine the limit below:

$$\lim_{x\to 0+} x \ln x$$

Hint. Rewrite the expression as a quotient and then use the L'Hôpital-Rule:

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} (-x) = 0$$

Problem 51

Find the following limit:

$$\lim_{x\to\infty} x^2 e^{-x}$$

Hint. First express the function in the form of a quotient, and then apply the L'Hôpital-Rule twice in a row:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

We can see from the procedure that the result remains true if the quadratic factor of x is replaced by any higher power of x. Indeed, by taking the derivative of the numerator sufficiently many times the power of x will disappear, while the denominator remains e^x in each step. Thus, for any integer n we have:

$$\lim_{x\to\infty} x^n e^{-x} = 0$$

Informally, this phenomenon can be interpreted by saying that e^x "tends faster to infinity" than any power of x.

Problem 52

How should we interpret the power 0^0? On the one hand, we may say it should be zero, because any power of zero is zero. But on the other hand, way may also say that it should be 1, because the "zero power of any number is 1". Which one is more reasonable?

We believe that we make the right decision if our choice makes the function

$$f(x) = x^x$$

continuous from the right at $x = 0$, that is there should not be a "gap" on the graph. (It cannot be continuous from the left, because the function is not even defined on the negative part of the real line.)

Find the right-hand limit of f at $x = 0$:

$$\lim_{x \to 0+} x^x = \lim_{x \to 0+} e^{x \ln x} = 1$$

since the limit of the exponent is 0 (we refer to an earlier problem).

ATTENTION! Here we relied on the continuity of e^x at 0, which is well known, since it is even differentiable at that point.

Therefore, the reasonable choice is 1.

Problem 53

Consider the function:

$$f(x) = x^2 - x - 1 + (x^2 - 5x + 6) \ln x$$

Does there exist a point $a \in [2, 3]$ so that $f'(a) = 4$?

Hint. Let us try to find a solution to the equation

$$f'(x) = 2x - 1 + (2x - 5) \ln x + \frac{1}{x}(x^2 - 5x + 6) = 4$$

in the interval $[2, 3]$, which is obviously impossible. This equation cannot be solved with algebraic manipulations.

However, if we realize that $f(2) = 1$ and $f(3) = 5$, then by Lagrange's Mean Value Theorem we can find a point $a \in [2, 3]$ for which

$$\frac{f(3) - f(2)}{3 - 2} = 4 = f'(a)$$

Thus, the equation has a solution in the interval $[2, 3]$.

ATTENTION!

Second solution. We could have argued the following way. Examine the derivative function at the endpoints of the interval. Then we find

$$f'(2) = 3 - \ln 2 < 4 \quad \text{moreover} \quad f'(3) = 5 + \ln 3 > 4$$

Since the derivative function is continuous, by Bolzano's Theorem there exists a point $a \in [2, 3]$, so that $f'(a) = 4$.

Problem 54

Find the monotone segments of the function

$$f(x) = x^2 e^{-x}$$

and find the extreme points (if they exist).

Hint. Calculate the derivative:

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x}$$

Obviously, we have two critical points: $x_1 = 0$ and $x_2 = 2$. By examining the sign of the we come to the following conclusion:

- if $-\infty < x < 0$, then $f'(x) < 0$, hence f is strictly monotone decreasing,
- if $0 < x < 2$, then $f'(x) > 0$, hence f is strictly monotone increasing,
- if $2 < x < +\infty$, then $f'(x) < 0$, hence f is strictly monotone decreasing.

As we see, the derivative changes the sign at both critical points, therefore:

- f has a (global) minimum at $x_1 = 0$,
- f has a local maximum at $x_2 = 2$.

It is easy to see that the maximum point $x_2 = 2$ is not global. In fact, the function is not bounded from above, since

$$\lim_{x \to -\infty} x^2 e^{-x} = +\infty$$

Incidentally, by the L'Hôpital-Rule we also have:

$$\lim_{x \to +\infty} x^2 e^{-x} = 0$$

(compare with an earlier exercise).

Problem 55
Find the convex and concave segments of the function above, and find the points of inflection.

Hint. The second derivative of the function is given by:

$$f''(x) = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = (x^2 - 4x + 2)e^{-x}$$

The zeros of the second derivative are:

$$x_1 = 2 - \sqrt{2} \quad \text{and} \quad x_2 = 2 + \sqrt{2}$$

By examining the sign of the second derivative we come to the conclusion:

- if $-\infty < x < 2 - \sqrt{2}$, then $f''(x) > 0$, and f is convex,
- if $2 - \sqrt{2} < x < 2 + \sqrt{2}$, then $f''(x) < 0$, and f is concave,
- if $2 + \sqrt{2} < x < +\infty$, then $f''(x) > 0$, and f is convex.

Therefore f'' changes the sign both at x_1, and x_2, and that means that both are inflection points of f.
ATTENTION! See the file Figures.pdf for the graph of the function!

Problem 56
The following function is defined with an unknown parameter a.

$$f(x) = 2ax + \ln x \quad x > 0$$

Determine the value of the parameter a so that the function has a local maximum at $x = 1/2$.

Hint. The derivative of the function is

$$f'(x) = 2a + \frac{1}{x}$$

The derivative is zero at the local maximum point, so at $x = 1/2$ we necessarily have

$$2a + 2 = 0$$

and this means $a = -1$. We have not finished yet, because we need to verify that the point $x = 1/2$ is really a maximum point. Take the second derivative:

$$f''(x) = -\frac{1}{x^2}$$

and this is negative everywhere, therefore the critical point is really a maximum point and $a = -1$ is the only solution to the problem.

Problem 57

Find the monotone segments of the following function:

$$f(x) = \frac{x}{1+x^3} \quad x \neq -1$$

Find the extreme points as well (if any).

Hint. The derivative function is:

$$f'(x) = \frac{1+x^3-3x^3}{(1+x^3)^2} = \frac{1-2x^3}{(1+x^3)^2}$$

As the denominator is positive, the sign of the derivative depends on the sign of the numerator. The summary is the following:

- if $-\infty < x < -1$, then $f'(x) > 0$, and f is strictly monotone increasing,
- if $-1 < x < 1/\sqrt[3]{2}$, then $f'(x) > 0$, and f is strictly monotone increasing,
- if $1/\sqrt[3]{2} < x < \infty$, then $f'(x) < 0$, and f is strictly monotone decreasing.

Since the derivative changes the sign at the critical point $x = 1/\sqrt[3]{2}$, the function possesses a local maximum at that point.

ATTENTION! At $x = 1/\sqrt[3]{2}$ the maximum is local, but not global! Indeed, at the point $x = -1$ is not defined, but its one-sided limits are:

$$\lim_{x\to -1-} \frac{x}{1+x^3} = +\infty \quad \text{and} \lim_{x\to -1+} \frac{x}{1+x^3} = -\infty$$

and consequently, the function is neither bounded from above, nor from below.

ATTENTION! Although the function f is strictly monotone increasing both on $(-\infty, -1)$ and on $(-1, 1/\sqrt[3]{2})$, that does not mean that f would be strictly monotone increasing on the interval $-\infty, 1/\sqrt[3]{2})$. In fact, the point $x = -1$ is not in the domain! See the remark above about the one-sided limits here!

Problem 58

Find the convex and concave segments, and the points of inflection of the function above.

Hint. Determine the second derivative for all $x \neq -1$:

$$f''(x) = \frac{-6x^2(1+x^3)^2 - (1-2x^3)\cdot 2(1+x^3)\cdot 3x^2}{(1+x^3)^4} = \frac{6x^2(x^3-2)}{(1+x^3)^3}$$

The second derivative has two roots: $x_1 = 0$ and $x_2 = \sqrt[3]{2}$. The signs are:

- if $-\infty < x < -1$, then $f''(x) > 0$, and f is convex,
- if $-1 < x < 0$, then $f''(x) < 0$, and f is concave,
- if $0 < x < \sqrt[3]{2}$, then $f''(x) < 0$, and f is concave,
- if $\sqrt[3]{2} < x < +\infty$, then $f''(x) > 0$, and f is convex.

As we can notice, at the point $x_1 = 0$ the second derivative does not change the sign, and hence, this is not a point of inflection. At the point $x_2 = \sqrt[3]{2}$ however, f'' changes the sign, so this a point of inflection.

ATTENTION!

Can we say that the function is concave on the whole interval $-1 < x < \sqrt[3]{2}$? See the comment to the previous problem, where the answer to a similar question was NO.

The answer here is YES, since the function is continuous on the whole interval, and in addition at every interior point we have $f''(x) \leq 0$.

See the graph of this function in the file Figures.pdf!

Problem 59

The function below is given with an unspecified parameter a.

$$f(x) = 2ax + \ln x \quad x > 0$$

Find the value of the parameter a such that the function is monotone decreasing on the open interval $(0, 1)$!

Hint. The derivative of the function is:

$$f'(x) = 2a + \frac{1}{x}$$

We are looking for a parameter a so that f is monotone decreasing on the open interval $(0, 1)$, which means

$$f'(x) = 2a + \frac{1}{x} \leq 0$$

for each $0 < x < 1$. If x is sufficiently close to the point 0, then the derivative can take arbitrarily large positive values, therefore, the derivative will be certainly positive regardless of the value of the parameter a. Thus, such a parameter does not exist.

Problem 60

Analyze the function

$$f(x) = xe^{-x^2}$$

and find the extreme points.

Hint. The derivative of the function is:

$$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1 - 2x^2)e^{-x^2}$$

As we can see, the function has two critical points:

$$x_1 = -\frac{1}{\sqrt{2}} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}$$

By examining the sign of the derivative, we obtain:

- if $-\infty < x < -1/\sqrt{2}$, then $f'(x) < 0$, and f strictly monotone decreasing,
- if $-1/\sqrt{2} < 1/\sqrt{2}$, then $f'(x) > 0$, and f is strictly monotone increasing,
- if $1/\sqrt{2} < x < +\infty$, then $f'(x) < 0$, and f is strictly monotone decreasing.

This gives us that x_1 is a minimum point, and x_2 is a maximum point, and in addition both are global!

ATTENTION!

We could have proceeded like recognizing that f is an odd function, i.e. it is symmetric with respect to the origin. Please verify, it is easy!

So we can focus our attention to the positive half line, anything that is a (global) maximum point, its negative on the negative side is a (global) minimum point. Please think about it!

Problem 61

Imagine that we consider the numbers

$$1, \quad \sqrt{2}, \quad \sqrt[3]{3}, \quad \sqrt[4]{4}, \quad \ldots \quad \sqrt[n]{n}, \quad \ldots$$

What do we think, which one of those is the largest?

As a matter of fact, is this a correct question? For infinetely many numbers there is not necessarily a largest!

Hint. Consider the sequence

$$a_n = \sqrt[n]{n}$$

On the one hand $a_1 = 1$, on the other hand, we have seen before that

$$\lim_{n \to \infty} a_n = 1$$

By the definition of this limit, we can say that there exists an index N, such that

$$1 < a_n < 1 + 0.01$$

if $n \geq N$. Therefore, there certainly exists a largest element in the sequence, because it can be selected from the first N (finitely many) elements. Thus, the question is correct.

Turn to the selection of the largest element. Consider the following function on the positive part of the real line:

$$f(x) = x^{1/x} \quad x > 0$$

At integer values this function takes the elements of the sequence. Since

$$f(x) = x^{1/x} = e^{\ln x/x}$$

then by the Chain-Rule:

$$f'(x) = \left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right) e^{\ln x/x} = \frac{1}{x^2}(1 - \ln x)e^{\ln x/x}$$

Obviously, the only critical point is $x = e$. The first and the last factors are always positive, so we deduce

- if $0 < x < e$, then $f'(x) > 0$, and f is strictly monotone increasing,
- if $e < x < +\infty$, then $f'(x) < 0$, and f is strictly monotone decreasing.

Therefore, the function f takes its global maximum at the point $x = e$. Since $e \approx 2,7182...$, we conclude that largest element of the sequence a_n can only be at $n = 2$ or $n = 3$. This is decided by direct comparison. Clearly

$$\sqrt{2} < \sqrt[3]{3}$$

so we have the largest element for $n = 3$.

Problem 62

Find the indefinite integral on the interval $x \geq -2$

$$\int \left(3x^2 - 4x + \sqrt{x+2}\right)\, dx$$

Hint. Integrate term by term:

$$\int \left(3x^2 - 4x + \sqrt{x+2}\right)\, dx = x^3 - 2x^2 + \frac{2}{3}(x+2)^{3/2} + C$$

Note that the square root means a power of 1/2.

Problem 63

Find the primitive function:

$$\int \sin 2x\, dx$$

Hint. By the duble angle formula $\sin 2x = 2 \sin x \cos x$ and this is precisely the derivative of $\sin^2 x$ (use the Chain-Rule), thus

$$\int \sin 2x\, dx = \sin^2 x + C$$

Second solution. We could use the formula

$$\left(-\frac{\cos 2x}{2}\right)' = \sin 2x$$

and hence

$$\int \sin 2x\, dx = -\frac{\cos 2x}{2} + C$$

ATTENTION!

We used two different procedures and came up with two different functions. Which one leads to the correct solution? Answer: both. Indeed, the functions differ only in an additive constant:

$$-\frac{\cos 2x}{2} = \frac{1}{2}\sin^2 x - \frac{1}{2}\cos^2 x = \sin^2 x - \frac{1}{2}\left(\sin^2 x + \cos^2 x\right) = \sin^2 x - \frac{1}{2}$$

Problem 64

Evaluate the definite integral:

$$\int_0^1 \frac{5x}{1+x^2}\,dx$$

Hint. Rewrite the integral this way:

$$\int_0^1 \frac{5x}{1+x^2}\,dx = \frac{5}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

In the latter integral the numerator is exactly the derivative of the denominator. In this case the fraction is the derivative of the logarithm of the denominator, that is:

$$\left(\ln(1+x^2)\right)' = \frac{2x}{1+x^2}$$

Verify this by using the Chain-Rule! The Newton-Leibniz-formula gives us

$$\int_0^1 \frac{5x}{1+x^2}\,dx = \frac{5}{2}\left[\ln(1+x^2)\right]_0^1 = \frac{5}{2}\ln 2$$

ATTENTION!

The above method can always be applied whenever $f(x) > 0$, and the integrand is of the form $f'(x)/f(x)$. In this case the primitive function is $\ln f(x)$. For example:

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx = -\int \frac{-2\sin x\cos x}{1+\cos^2 x}\,dx = -\ln(1+\cos^2 x) + C$$

Problem 65

Another useful observation is (assume that $n \neq -1$):

$$\int f(x)^n \cdot f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + C$$

Verify it by exploiting the Chain-Rule! For instance:

$$\int 3x(5+x^2)^3\,dx = \frac{3}{2}\int 2x(5+x^2)^3\,dx = \frac{3}{8}(5+x^2)^4 + C$$

or very similarly in the definite integral below:

$$\int_0^2 2x^2\sqrt{1+x^3}\,dx = \frac{2}{3}\int_0^2 3x^2\sqrt{1+x^3}\,dx = \frac{2}{3}\cdot\frac{2}{3}\left[(1+x^3)^{3/2}\right]_0^2 = \frac{4}{9}(27-1) = \frac{104}{9}$$

by the Newton-Leibniz-formula.

Problem 66

Carry out integration by parts in the problem below:

$$\int_0^\pi x^2\cos x\,dx$$

Hint. Obviously, the correct allocation of roles is $f'(x) = \cos x$, and $g(x) = x^2$, then

$$\int_0^\pi x^2\cos x\,dx = \left[x^2\sin x\right]_0^\pi - \int_0^\pi 2x\sin x\,dx$$

We clearly have zero in the brackets. On the right-hand side use integration by parts again:

$$\int_0^\pi 2x \sin x \, dx = [-2x \cos x]_0^\pi + \int_0^\pi 2 \cos x \, dx = 2\pi$$

because the last integral is zero. Finally

$$\int_0^\pi x^2 \cos x \, dx = -2\pi$$

Problem 67

Evaluate the definite integral:

$$\int_0^1 \arctan x \, dx$$

Hint. We use integration by parts in a smart way: set $f'(x) = 1$ and $g(x) = \arctan x$. Then the function $\arctan x$ has to be differentiated instead of integrating. Therefore,

$$\int_0^1 1 \cdot \arctan x \, dx = [x \arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx = \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+x^2) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

A very similar trick is applied for finding the primitive function of $\arcsin x$

$$\int 1 \cdot \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x + \sqrt{1-x^2} + C$$

in the open interval $(-1, 1)$. In the last integral we used the fact that

$$-\frac{x}{\sqrt{1-x^2}}$$

is precisely the derivative of $\sqrt{1-x^2}$. Verify this!
In an analogous way we can find the primitive function of $\arccos x$ as well. Homework!

Problem 68

Use substitution in the following problem:

$$\int 5t^3 \sqrt{2+t^4} \, dt$$

Hint. Introduce the substitution: $x = g(t) = 2 + t^4$. Then $g'(t) = 4t^3$. Set $f(x) = \sqrt{x}$, so:

$$\begin{aligned} \int 5t^3 \sqrt{2+t^4} \, dt &= \frac{5}{4} \int 4t^3 \sqrt{2+t^4} \, dt = \frac{5}{4} \int g'(t) \sqrt{g(t)} \, dt \\ &= \frac{5}{4} \int \sqrt{x} \, dx = \frac{5}{6} x^{3/2} + C = \frac{5}{6} (2+t^4)^{3/2} + C \end{aligned}$$

where we performed the backsubstitution as well.

Problem 69

Consider now a substitution in a definite integral:

$$\int_0^{\pi/2} \frac{\cos t}{1+\sin^2 t} \, dt$$

Hint. The convenient substitution in this integral is:

$$x = g(t) = \sin t \quad g'(t) = \cos t \quad \text{and} \quad f(x) = \frac{1}{1+x^2}$$

Carry out the substitution:

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos t}{1+\sin^2 t} \, dt &= \int_0^{\pi/2} g'(t) f(g(t)) \, dt = \int_0^{\pi/2} \frac{g'(t)}{1+g(t)^2} \, dt \\ &= \int_0^1 f(x) \, dx = \int_0^1 \frac{1}{1+x^2} \, dx = [\arctan x]_0^1 = \frac{\pi}{4} \end{aligned}$$

Observe how we changed the bounds of the integration!

Problem 70

Take a look at a substitution the other way:

$$\int_0^1 x\sqrt{1+x}\,dx$$

Hint. Apply the setting $x = g(t) = t - 1$, then $g'(t) = 1$, and hence

$$\int_0^1 x\sqrt{1+x}\,dx = \int_1^2 (t-1)\sqrt{t}\,dt = \int_1^2 \left(t^{3/2} - t^{1/2}\right)\,dt = \left[\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2}\right]_1^2$$

that we can evaluate by plugging in the bounds (skipped). Please observe again, how the bounds of the integration are changed!

Problem 71

Find the solution to the linear differential equation

$$\begin{aligned} y' &= -2y + 2 \\ y(0) &= 1 \end{aligned}$$

(also called initial value problem).

Hint. Use the Cauchy-formula:

$$y(t) = e^{-2t}\left(1 + \int_0^t 2e^{2s}\,ds\right) = e^{-2t}\left(1 + \left[e^{2s}\right]_0^t\right) = 1$$

So the only solution to the problem is the constant function $y = 1$.

Problem 72

We can extend the solution method to linear differential equations with non-constant coefficients the following way.

Suppose that a and b are continuous functions on the interval I, and consider the initial value problem

$$\begin{aligned} y' &= a(t)y + b(t) \\ y(t_0) &= y_0 \end{aligned}$$

where $t_0 \in I$ and $y_0 \in \mathbb{R}$ are given. Introduce the notation

$$A(t) = \int_{t_0}^t a(s)\,ds$$

and multiply the equation by the function $e^{-A(t)}$. Rearrenging the terms we get

$$y'e^{-A(t)} - a(t)e^{-A(t)}y = \left(ye^{-A(t)}\right)' = b(t)e^{-A(t)}$$

Integrate both sides and keep an eye on the initial condition:

$$y(t)e^{-A(t)} - y_0 = \int_{t_0}^t b(s)e^{-A(s)}\,ds$$

since the function A is zero at t_0. Isolate y on the left-hand side:

$$y(t) = e^{A(t)}\left(y_0 + \int_{t_0}^t b(s)e^{-A(s)}\,ds\right)$$

We call this equality the (extended) Cauchy-formula.

Problem 73

By using the Cauchy-formula find the solution to the initial value problem

$$\begin{aligned} y' &= \frac{y}{t} + 2t^2 \\ y(1) &= 2 \end{aligned}$$

on the positive half line $t > 0$.

Hint. In this situation $t_0 = 1$, $y_0 = 2$, moreover $a(t) = 1/t$ and $b(t) = 2t^2$. Furthermore

$$A(t) = \int_1^t \frac{1}{s}\, ds = \ln t$$

and accordingly

$$e^{A(t)} = t \qquad \text{and} \qquad e^{-A(t)} = \frac{1}{t}$$

Substituting all these into the Cauchy-formula, we obtain

$$y(t) = t\left(2 + \int_1^t 2s\, ds\right) = t(t^2+1) = t^3 + t$$

on the positive half line. Verify this result by differentiation and direct substitution!

Problem 74

Evaluate the improper integral

$$\int_0^\infty \frac{1}{1+x^2}\, dx$$

Hint. For any $b > 0$ we have:

$$\int_0^b \frac{1}{1+x^2}\, dx = [\arctan x]_0^b = \arctan b$$

The limit of the arc tangent function when $b \to +\infty$ is $\pi/2$, therefore

$$\int_0^\infty \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$$

The integrand is an even function, so we can immediately come up with the following equality :

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = \pi$$

Problem 75

Take an integer $n \in \mathbb{N}$ and evaluate the following improper integral:

$$I_n = \int_0^\infty x^n e^{-x}\, dx$$

Attention, we have evaluated this integral for $n = 0$, $n = 1$ and $n = 2$ in the lecture!

Hint. Referring to the lectures, we have seen that $I_0 = 1$, $I_1 = 1$, and $I_2 = 2$. Now take an integer $n > 2$, and use integration by parts with the settings $f'(x) = e^{-x}$, and $g(x) = x^n$ respectively. Then

$$I_n = \int_0^\infty x^n e^{-x}\, dx = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

The value of the brackets is zero. Think about it: at the lower bound x^n is zero, while at the upper bound we get zero in view of the L'Hôpital-Rule. On the other hand, the integral on the right-hand side is precisely I_{n-1}. Thus, for every integer n we have

$$I_n = n \cdot I_{n-1}$$

Since $I_1 = 1$, this implies that $I_n = n!$.

Problem 76

In a similar way to the preceding problem, find the improper integral

$$I_n = \int_0^\infty \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}\, dx$$

where $\lambda > 0$ is a given constant.

Hint. Integrate by parts, now with the settings $f'(x) = x^{n-1}$, and $g(x) = e^{-\lambda x}$. Then we get

$$\begin{aligned} I_n &= \int_0^\infty \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}\, dx = \frac{\lambda^n}{(n-1)!}\left[\frac{x^n}{n} e^{-\lambda x}\right]_0^\infty + \frac{\lambda^n}{(n-1)!}\int_0^\infty \frac{x^n}{n}\lambda e^{-\lambda x}\, dx \\ &= \frac{\lambda^{n+1}}{n!}\int_0^\infty x^n e^{-\lambda x}\, dx = I_{n+1} \end{aligned}$$

since (just like in the previous problem) the value of the brackets is zero. On the other hand, for $n = 1$ we have

$$I_1 = \int_0^\infty \lambda e^{-\lambda x}\, dx = 1$$

see the lectures!. Therefore, for any integer $n \in \mathbb{N}$ we get $I_n = 1$.

ATTENTION!

Carry out the integration by parts in a reverse order, and make sure that we receive precisely the same answer. We note that this result has an important application in probability theory.

Problem 77

Is the following improper integral convergent?

$$\int_1^\infty \frac{\sqrt{x}}{x + x^2}\, dx$$

Hint. We can give an upper estimate on the integrand on the interval $[1, \infty)$ in the following way:

$$\frac{\sqrt{x}}{x + x^2} < \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

Then for an arbitrary $b > 1$ we obtain

$$\int_1^b \frac{\sqrt{x}}{x + x^2}\, dx \le \int_1^b \frac{1}{x^{3/2}}\, dx$$

The integral on the right-hand side is convergent when $b \to \infty$, because in the denominator the exponent of x is greater than 1, and the limit of the integral is 2 (see the lectures). Hence, the integral in the original problem is convergent, and

$$\int_1^\infty \frac{\sqrt{x}}{x + x^2}\, dx \le 2$$

In a similar way we can find a lower estimate as well:

$$\frac{\sqrt{x}}{x + x^2} \ge \frac{\sqrt{x}}{x^2 + x^2} = \frac{1}{2}\cdot\frac{1}{x^{3/2}}$$

whenever $x \ge 1$, and consequently

$$\int_1^\infty \frac{\sqrt{x}}{x + x^2}\, dx \ge \frac{1}{2}\cdot\int_1^\infty \frac{1}{x^{3/2}}\, dx = 1$$

This inequality combined with the upper estimate yields

$$1 \le \int_1^\infty \frac{\sqrt{x}}{x + x^2}\, dx \le 2$$

Second solution. Carry out the substitution $x = g(t) = t^2$ (where $t \geq 1$). Then $g'(t) = 2t$, and for any $b > 1$ we obtain

$$\begin{aligned}\int_1^b \frac{\sqrt{x}}{x+x^2}\,dx &= \int_1^{\sqrt{b}} \frac{2t^2}{t^2+t^4}\,dt = 2\int_1^{\sqrt{b}} \frac{1}{1+t^2}\,dt \\ &= 2\left[\arctan t\right]_1^{\sqrt{b}} = 2\left(\arctan\sqrt{b} - \frac{\pi}{4}\right)\end{aligned}$$

(Please note the change of the bounds!) When passing to the limit $b \to \infty$ the limit of the arc tangent function is $\pi/2$. Therefore, the integral is convergent, and

$$\int_1^\infty \frac{\sqrt{x}}{x+x^2}\,dx = \frac{\pi}{2}$$

ATTENTION!

The first method to verify the convergence of the improper integral is much easier. Its disadvantage is, however that it only provides an estimate and no exact value!

Problem 78

Find the sum of the following series:

$$\sum_{k=1}^\infty k\left(\frac{5}{6}\right)^k$$

Hint. For every $-1 < x < 1$ the sum of the geometric series is

$$\sum_{k=0}^\infty x^k = \frac{1}{1-x}$$

Taking the derivative of both sides, we have

$$\sum_{k=1}^\infty kx^{k-1} = \frac{1}{(1-x)^2}$$

(since for $k = 0$ the derivative of the constant term is zero). By substituting $x = 5/6$ we get

$$\sum_{k=1}^\infty k\left(\frac{5}{6}\right)^k = \frac{5}{6}\sum_{k=1}^\infty k\left(\frac{5}{6}\right)^{k-1} = \frac{5}{6}\cdot\frac{1}{(1-5/6)^2} = 30$$

Problem 79

Find the sum function of the power series

$$\sum_{k=1}^\infty k^2x^k$$

in the open interval $-1 < x < 1$.

Hint. Compute the second derivative of the geometric series in the given interval:

$$\sum_{k=2}^\infty k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$$

This is almost what we want, but we have a multiplier $k(k-1)$ instead of k^2, and both sides have to be multiplied by x^2. Our original problem is decomposed into a sum like this:

$$\begin{aligned}\sum_{k=1}^\infty k^2x^k &= x^2\sum_{k=2}^\infty k(k-1)x^{k-2} + x\sum_{k=1}^\infty kx^{k-1} \\ &= x^2\frac{2}{(1-x)^3} + x\frac{1}{(1-x)^2} = \frac{x^2+x}{(1-x)^3}\end{aligned}$$

since for $k = 1$ we have $k^2 = 1$ as well.

ATTENTION!

Make sure that all these power series are convergent in the open interval $(-1,1)$.

Problem 80

Determine the sum function of the power series:

$$f(x)=\sum_{k=1}^{\infty}2^{k-1}\frac{x^{k+1}}{k!}$$

Hint. Factor out x, and rewrite the series in the following form:

$$f(x)=\frac{x}{2}\sum_{k=1}^{\infty}2^{k}\frac{x^{k}}{k!}=\frac{x}{2}\sum_{k=1}^{\infty}\frac{(2x)^{k}}{k!}$$

The latter is almost the Taylor-series of the function e^{2x}. However, the summation starts from $k=1$, therefore the term that belongs to $k=0$ is missing. Thus, we have

$$f(x)=\frac{x}{2}\left(e^{2x}-1\right)$$

where inside the parentheses we subtracted the missing term.

Problem 81

Find the sum of the power series:

$$\sum_{k=0}^{\infty}k\frac{x^{k}}{k!}$$

(We will refer to this sum in Probability Theory.)

Hint. Simplify by k and factor out x:

$$\sum_{k=0}^{\infty}k\frac{x^{k}}{k!}=x\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}=xe^{x}$$

Please observe that the last series is exactly the Taylor-series of e^{x} (by a shift of indexing). This power series is convergent on the entire real line.

Problem 82

Find the Taylor-series of the function $f(x)=\sin x$.

Hint. On the one hand $f(0)=0$, on the other hand the even order derivatives of the sine function are plus or minus sine, all of them are zero at the origin. That means

$$f^{(2k)}(0)=0 \qquad k=0,1,2,3,\ldots$$

The odd order derivatives are plus or minus cosine, in particular:

$$f^{(4k+1)}(0)=\cos 0=1 \quad \text{and} \quad f^{(4k+3)}(0)=-\cos 0=-1 \qquad k=0,1,2,3,\ldots$$

As a consequence, the Taylor-series will have the form:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^{k} &= \sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)!}x^{2k+1}\\ &= x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots\end{aligned}$$

Verify that this power series is convergent on the whole real line!

It can also be proven that the sum function is $\sin x$, that is

$$\sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\ldots$$

The proof goes beyond the scope of this semester.

Problem 83

Find the Taylor-series of the function $f(x) = \cos x$.

Hint. On the one hand $f(0) = 1$, on the other hand the odd order derivatives of the cosine function are plus or minus sine, all of them are zero at the origin. That means

$$f^{(2k+1)}(0) = 0 \qquad k = 0, 1, 2, 3, \dots$$

The even order derivatives are plus or minus sine, in particular:

$$f^{(4k)}(0) = \cos 0 = 1 \quad \text{and} \quad f^{(4k+2)}(0) = -\cos 0 = -1 \qquad k = 0, 1, 2, 3, \dots$$

As a consequence, the Taylor-series will have the form:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

Verify that this power series is convergent on the whole real line!
It can also be proven that the sum function is $\cos x$, that is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The proof goes beyond the scope of this semester.

Problem 84

Find the power series whose sum function is $f(x) = \arctan x$.

Hint. Consider the geometric series whose ratio is $-x^2$, where $-1 < x < 1$. In this case the the series is convergent, since $-1 < -x^2 \leq 0$, and its sum is given by:

$$\sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2}$$

Integrate both sides from zero to x (ATTENTION! The power series can be integrated term by term, this is far from being trivial!):

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = \int_0^x \frac{1}{1+t^2}\, dt = \arctan x$$

This series is convergent in the open interval $(-1, 1)$, PLEASE VERIFY!
At $x = 1$ we obtain a series with alternating signs. Completely analogously to our earlier example (see Chapter 2 for details) we can show that this series is convergent. This leads to the celebrated identity:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

(taking into account that $\arctan 1 = \pi/4$).

Problem 85

Find the critical points of the function:

$$f(x, y) = x^2 - 2x + 5 + y^2 e^{-y}$$

Hint. Calculate the partial derivatives and examine the system of equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 2 = 0 \\ \frac{\partial f}{\partial y} &= 2ye^{-y} - y^2 e^{-y} = 0 \end{aligned}$$

The solutions are $x = 1$, moreover $y = 0$ and $y = 2$. Therefore, the function has two critical points, and they are $P(1, 0)$ and $Q(1, 2)$.

It is easy to see that $P(1, 0)$ is a minimum point of the function, since at $x = 1$ we have a global minimum of the first three terms that depend only on x. On the other hand, at $y = 0$ we have a global minimum of the last term that depends only on y. Consequently, for any point $(x, y) \neq (1, 0)$ we get

$$f(x, y) > f(1, 0)$$

However, the other critical point $Q(1, 2)$ is not an extremum. Indeed, if a coordinate $x \neq 1$ is taken, then $f(x, 2) > f(1, 2)$. On the other hand, if pick a coordinate $y \neq 2$ for example in the interval $[1, 3]$, then we have $f(1, y) < f(1, 2)$. Thus, $Q(1, 2)$ is neither a minimum point, nor a maximum point.

Problem 86

Consider the function

$$f(x, y) = \sqrt{1 + x^2 + 2y^2} \quad \text{where} \quad x = g_1(t) = te^{-t} \quad \text{and} \quad y = g_2(t) = e^{-2t}$$

and find the derivative of the composition function $F(t) = f(g_1(t), g_2(t))$ by applying the Chain Rule.

Hint. Find the partial derivatives of f:

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1 + x^2 + 2y^2}} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2y}{\sqrt{1 + x^2 + 2y^2}}$$

Then calculate the derivatives of g_1 and g_2:

$$g_1'(t) = (1 - t)e^{-t} \qquad \text{and} \qquad g_2'(t) = -2e^{-2t}$$

Now apply the Chain Rule:

$$\begin{aligned} F'(t) &= \frac{\partial f}{\partial x}(g_1(t), g_2(t))g_1'(t) + \frac{\partial f}{\partial y}(g_1(t), g_2(t))g_2'(t) \\ &= \frac{te^{-t}}{\sqrt{1 + t^2e^{-2t} + 2e^{-4t}}}(1 - t)e^{-t} - \frac{2e^{-2t}}{\sqrt{1 + t^2e^{-2t} + 2e^{-4t}}}2e^{-2t} \end{aligned}$$

For practicing also find the derivative of F by a direct substitution of the functions g_1 and g_2, and by differentiating this composition function (it is going to be somewhat complicated!). Make sure that you get the same result.

Problem 87

Consider the function below, and find its tangent plane at the point $P(1, 1)$

$$f(x, y) = (4x^2 - 2y^3)\sqrt{x^2 + y^2 + 2}$$

Hint. First calculate the partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 8x\sqrt{x^2 + y^2 + 2} + (4x^2 - 2y^3)\frac{x}{\sqrt{x^2 + y^2 + 2}} \\ \frac{\partial f}{\partial y} &= -6y^2\sqrt{x^2 + y^2 + 2} + (4x^2 - 2y^3)\frac{y}{\sqrt{x^2 + y^2 + 2}} \end{aligned}$$

At the given point $P(1, 1)$ we have

$$\frac{\partial f}{\partial x}(1, 1) = 17 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(1, 1) = -11$$

Direct substitution shows that $f(1, 1) = 4$. Thus, the equation of the tangent plane is:

$$17(x - 1) - 11(y - 1) = z - 4$$

and this equation could be further simplified.

Problem 88

Find the value of the parameter a, if the tangent plane to the function

$$f(x,y) = ay\sqrt{4+x^2+y^2}$$

at the point $x=2,\ y=1$ passes through the point $P(3;2;13)$.

Hint. First find the (parametric) equation of the tangent plane. We need the partial derivatives of f that are:

$$\begin{aligned} \frac{\partial f}{\partial x} &= ay\frac{x}{\sqrt{4+x^2+y^2}} \\ \frac{\partial f}{\partial y} &= a\sqrt{4+x^2+y^2} + ay\frac{y}{\sqrt{4+x^2+y^2}} \end{aligned}$$

The values of the partial derivatives at the given point are:

$$\frac{\partial f}{\partial x}(2,1) = \frac{2a}{3} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(2,1) = 3a + \frac{a}{3} = \frac{10a}{3}$$

Direct substitution shows that $f(2,1) = 3a$. Hence, the (parametric!) equation of the tangent plane is:

$$\frac{2a}{3}(x-2) + \frac{10a}{3}(y-1) = z - 3a$$

If this plane passes through the point $P(3,2,13)$, then the coordinates of the point must satisfy the equation of the plane:

$$\frac{2a}{3} + \frac{10a}{3} = 13 - 3a$$

This equation has a single solution $a = 13/7$, which is the solution of the problem.

Problem 89

Find the critical points of the function:

$$f(x,y) = \frac{2}{x} + \frac{1}{y} + \frac{xy}{16}$$

Hint. Calculate the partial derivatives and examine the following system of equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= -\frac{2}{x^2} + \frac{y}{16} = 0 \\ \frac{\partial f}{\partial y} &= -\frac{1}{y^2} + \frac{x}{16} = 0 \end{aligned}$$

The unique solution to this system is $x = 4$ and $y = 2$.

Now we can argue the following way. If $y = 2$ is fixed, the function

$$\frac{2}{x} + \frac{1}{2} + \frac{x}{8}$$

has a minimum at $x = 4$. Similarly, if $x = 4$ is fixed, then the function

$$\frac{1}{2} + \frac{1}{y} + \frac{y}{4}$$

has a minimum at $y = 2$. Verify both statements by differentiation!

These arguments lead us to believe that the function has a minimum at $x = 4,\ y = 2$.

ATTENTION!

This is not a proof! It might happen that the function does not have an extremum. Starting from the critical point $P(4,2)$ we only moved the coordinate x alone, or the coordinate y alone!

In order to prove that $P(4,2)$ is really a minimum point, consider the function

$$f(x,y) = \frac{2}{x} + \frac{1}{y} + \frac{xy}{16}$$

in the positive quadrant, where $x > 0$ and $y > 0$. Direct substitution shows that $f(4, 2) = 3/2$. On the other hand, all three terms on the right-hand side are positive.

For three pieces of positive numbers a_1, a_2 and a_3 the inequality for arithmetic-geometric means gives us

$$\frac{a_1 + a_2 + a_3}{3} \geq \sqrt[3]{a_1 a_2 a_3}$$

where equality holds if and only is $a_1 = a_2 = a_3$. Now apply this inequality for the numbers

$$a_1 = \frac{2}{x}, \qquad a_2 = \frac{1}{y} \qquad \text{és} \qquad a_3 = \frac{xy}{16}$$

After substitution and multiplication by 3 we obtain

$$\frac{2}{x} + \frac{1}{y} + \frac{xy}{16} \geq 3 \cdot \sqrt[3]{\frac{1}{8}} = \frac{3}{2}$$

This exactly means that $f(x, y) \geq f(4, 2)$ in the positive quadrant, thus $P(4, 2)$ is really a minimum point.

Even further, in the inequality for arithmetic-geometric means equality holds if and only if

$$\frac{2}{x} = \frac{1}{y} = \frac{xy}{16}$$

The only solution to this system of equations is $x = 4$ and $y = 2$. Therefore, for every point $(x, y) \neq (4, 2)$ in the positive quadrant

$$f(x, y) > f(4, 2)$$

We conclude that $P(4, 2)$ is a strict global minimum point of f in the positive quadrant. (Remark that this last result was exclusively based on the inequality for arithmetic-geometric means. So, the problem can be solved even without differentiation.)

Problem 90

Determine the value of the parameter a, if the tangent plane to the function

$$f(x, y) = axy\sqrt{4 + x^2 + y^2} - 9$$

at $P(2, 1)$ passes through the point $Q(4, 0, 3)$.

Hint. First find the partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= ay\sqrt{4 + x^2 + y^2} + axy\frac{x}{\sqrt{4 + x^2 + y^2}} \\
\frac{\partial f}{\partial y} &= ax\sqrt{4 + x^2 + y^2} + axy\frac{y}{\sqrt{4 + x^2 + y^2}}
\end{aligned}$$

At the given point $P(2, 1)$ we have

$$\frac{\partial f}{\partial x}(2, 1) = 3a + \frac{4a}{3} = \frac{13a}{3} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(2, 1) = 6a + \frac{2a}{3} = \frac{20a}{3}$$

Direct substitution shows that $f(2, 1) = 6a - 9$. Hence the (parametric!) equation of the tangent plane:

$$\frac{13a}{3}(x - 2) + \frac{20a}{3}(y - 1) = z - 6a + 9$$

If this plane passes through $Q(4, 0, 3)$, then the coordinates of the point must satisfy the equation of the tangent plane:

$$\frac{26a}{3} - \frac{20a}{3} = 12 - 6a$$

The only solution to this equation is $a = 3/2$, which is the solution to the problem.

Problem 91

Decide whether the statement is TRUE or FALSE.

- If $a_n \to 0$ and b_n are sequences, then $a_n b_n \to 0$.
- The tangent line to the graph of the function $f(x) = 2x \ln x$ at $x = 1$ passes through the point $P(3; 6)$.
- If f is continuous on the interval $[0, 1]$, then takes its minimum in the interval.
- A function is continuous on the interval $[0, 1]$, if it has a limit at every point.
- Suppose a function f is differentiable in the open interval (a, b). The function f has a minimum at an interior point x_0 if and only if $f'(x_0) = 0$.
- Suppose f is twice differentiable in the open interval (a, b). Then f is concave if and only if $f''(x) \leq 0$ at every point in the interval.
- If f is differnetiable at the point x_0, then f is continuous at x_0.
- If f is strictly monotone increasing on the real line, then at every point $f'(x) > 0$.
- If the sequence b_n is bounded, and $a_n \to 0$, then $\lim_{n\to\infty} a_n b_n = 0$.
- If f is twice differentiable, and x_0 is a local maximum point, then $f'(x_0) = 0$ és $f''(x_0) < 0$.
- If f is continuous on the interval $[0, 1]$, then it is bounded on $[0, 1]$.
- The tangent line to the graph of the function $f(x) = -2x + 4 \ln x$ at $x = 1$ passes through the point $P(3; -2)$.
- A function is continuous on $[0, 1]$ if and only if it has a left-hand limit and right-hand limit at every point of the interval, and two limits coincide.
- A sequence is convergent if and only if it is monotone and bounded.
- A series with positive terms is convergent if and only if the sequence of partial sums is bounded.
- The sum function of a power series is continuous in the interval of convergence.
- If $|f|$ is continuous on the interval $[a, b]$, then so is f.