Probability course, Exercises

The numbering refers to the chapters of the online lectures

Chapter 13

Problem 1

The number of permutations (arrangements) of n different objects is $n!$. However, if we have n_1 objects of one kind, n_2 objects of a second kind, and so forth n_k of a k-th kind that are identical from the arrangements point of view, then the number of different permutations is:

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

For example, we write the digits 0, 1, 1, 2, 3, 3, 3, 4 on eight pieces of paper. Using all eight pieces, how many 8-digit numbers can be created?

Hint. Counting only the different permutations we have

$$\frac{8!}{2! \cdot 3!} - \frac{7!}{2! \cdot 3!}$$

where we discarded those that start with the digit 0 (those are not 8-digit numbers).

Problem 2

Five married couples bought 10 tickets for a concert in one row, next to each other. How many ways can they be seated

(a) with no restriction?

(b) if couples are to sit together?

(c) if the men sit together and the women sit together?

(d) if men and women are alternating?

Hint. The solutions are:

$$(a): 10! \quad (b): 5! \cdot 2^5 \quad (c): 2 \cdot 5! \cdot 5! \quad (d): 2 \cdot 5! \cdot 5!$$

Problem 3

Eight people line up to get on a bus. In how many ways can they be arranged

(a) with no restriction?

(b) if 3 people insist on staying together?

(c) if 2 people refuse to follow each other?

Hint. The solutions are:

$$(a): 8! \quad (b): 3! \cdot 6! \quad (c): 8! - 2! \cdot 7!$$

Problem 4

In the game of Bridge the 52 cards of a deck of playing cards are evenly distributed among the 4 players (each player gets 13 cards). How many deals are possible?

Hint. First solution. We imagine the deal as follows. The 52 cards are arranged in a row (anyhow), we have 52! such permutations. Then the first 13 cards go the first player, the next 13 cards go to the second player, and so forth. For a player it is completely irrelevant in which order he receives the cards, it only counts what he holds in his hand. Therefore, from the arrangements point of view, the cards held by one player should be regarded as identical (of one kind). Thus, the solution of this permutation problem is:

$$\frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}$$

Second solution. Now the deal goes this way. The dealer selects 13 cards from the deck of 52 cards, and gives them to the first player, then the dealer selects another 13 cards, and gives them to the second player, and so forth, finally the remainig 13 cards are given to the fourth player. Each step is a problem of combinations, hence, the solution is:

$$\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13}$$

(since we have no choice for the fourth player).

ATTENTION!

The two approaches use two totally different combinatorial methods: permutations and combinations. Which one is correct? Answer: both! Please verify this directly by expanding the binomial coefficients!

Problem 5

We toss a pair of identical (indistinguishable) playing dice. How many outcomes are possible?

Hint. Since the dice are indistinguishable, we observe the outcomes $(2, 1)$ and $(1, 2)$ as identical. Find out how many outcomes are there with two different numbers, and then add the number of outcomes with two identical numbers. Then the number of observable outcomes is:

$$\binom{6}{2} + 6 = 21$$

We could have argued the following way: we need to select two numbers out of 6, and the order does not count, and each number can be selected twice. The solution of this combination problem is:

$$\binom{6+2-1}{2} = \binom{7}{2} = 21$$

Problem 6

The Binomial Theorem states that if a and b are real numbers, and n is any integer, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

This identity can be justified relatively easily, if we expand the n-th power as a product of n factors. If an integer k is fixed, then how many terms of the form $a^k b^{n-k}$ will be in the expanded product? Answer: as many as we can select k factors out of n from which we take the multiplier a, and we choose the multiplier b from the remaining factors. The number of these selections is the binomial coefficient

$$\binom{n}{k}$$

Obviously, we have

$$\binom{n}{k} = \binom{n}{n-k}$$

Based on the Binomial Theorem we can easily verify the identities below:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

or for instance

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (2+1)^n = 3^n$$

moreover

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1-1)^k = 0$$

Problem 7

Prove the following identity:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Hint. First solution. By taking the common denominator of the two fractions:

$$\begin{aligned}\binom{n}{k}+\binom{n}{k+1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k+1)!(n-k-1)!}=\frac{(k+1)n!+(n-k)n!}{(k+1)!(n-k)!}\\ &= \frac{(n+1)n!}{(k+1)!((n+1)-(k+1))!}=\binom{n+1}{k+1}\end{aligned}$$

and the identity is proven.

Second solution. Suppose that in a bowl there are n apples and 1 pear. In how many ways can we select $k+1$ pieces of fruit from the bowl (apples or pear)? On the one hand it is obvious that the number of selections is

$$\binom{n+1}{k+1}$$

Now we examine if the pear is in the selection or it is not. Below the first term is the number of selections where pear is not selected, the second term is where the pear is selected:

$$\binom{n}{k+1}+\binom{n}{k}$$

The two expressions must be equal, and this way the identity is proven.

ATTENTION! You should recognize that we proved an identity by creating a combinatorial problem, and solved it two ways. One way gives the left-hand side, the other way gives the right-hand side. There is no need for common denominator and complicated calculations!

Problem 8

Prove the following identity:

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\ldots+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}$$

Hint. We can immediately see that finding the common denominator (like in the first solution of the previous problem) is pretty much hopeless. Try to use the "apple-pear" approach of the second solution instead.

There are n apples and m pears in a bowl. In how many ways can we select k pieces of fruit from the bowl? On the one hand, the answer is obviously

$$\binom{n+m}{k}$$

On the other hand, we may want to keep track of the number of apples and pears in the sample. Going through all possible options, we get:

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\ldots+\binom{n}{k}\binom{m}{0}=\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}$$

The two formulae provide the solution to the same problem. Thus, the identity is verified.

Problem 9

In an Economics class the probability that a randomly selected student passed the Math exam is 0.72, passed the Microeconomics exam is 0.64, and passed both is 0.54. Find the probability that

(a) the student passed at least one of the exams.

(b) the student passed the Math exam, but did not pass the Microeconomics exam.

(c) the student passed none of the exams.

Hint. Let A denote the event that the student passed the Math exam, and let B denote the event that the student passed the Microeconomics exam. Then $P(A)=0.72$ and $P(B)=0.64$, moreover $P(A\cap B)=0.54$. Using this terminology, the solutions are:

(a) $P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.82$

(b) $P(A\cap\overline{B})=P(A)-P(A\cap B)=0.72-0.54=0.18$

(c) $P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})=1-P(A\cup B)=1-0.82=0.18$

where in the solution of (c) we used the De Morgan-formula.

Chapter 14

Problem 10

From a deck of 52 playing cards we select 5 cards at random. Find the probability that we select 2 spades and 3 diamonds.

Hint. There are 13 cards in each suit, therefore:

$$\frac{\binom{13}{2}\binom{13}{3}}{\binom{52}{5}}$$

Problem 11

From a deck of 52 playing cards we select 5 cards at random, without replacement. What is the probability that all 4 suits (clubs, diamonds, hearts, spades) are represented in the sample?

Hint. Examine the following argument. Let A denote the event that all 4 suits appear in the sample of 5 cards. Since the choice of any 5 cards is equally likely, we deal with a classical probability space.

In order to find out the number of favorable outcomes, take into account that we have 13 options for each suit. Once one card from each suit has been taken, then any card can be chosen from the remaining 48 cards.

The total number of outcomes: as many as the number of selections of 5 cards out of 52. So:

$$P(A) = \frac{13^4 \cdot 48}{\binom{52}{5}}$$

Is this the correct solution? If not, how could it be fixed?

This formula is NOT correct, because in the numerator the selections

$$\spadesuit A, \heartsuit A, \diamondsuit A, \clubsuit A, \spadesuit K$$

and

$$\spadesuit K, \heartsuit A, \diamondsuit A, \clubsuit A, \spadesuit A$$

are both counted, although they represent the same selection.

The right argument is the following. If each suit appears in the sample, then one suit will be duplicated. To select that suit we have 4 options and we draw 2 cards from the duplicated suit, and 1 from the others. Thus, the correct probability is:

$$P(A) = \frac{4 \cdot \binom{13}{2} \cdot \binom{13}{1}^3}{\binom{52}{5}}$$

which is precisely one half of the wrong solution.

ATTENTION!

In most combinatorial problems, counting some cases twice is a much more frequent mistake, than missing some cases. If two approaches yield two different solutions, then most of the time the bigger number should be suspicious!

Problem 12

On a seasonal sale in a supermarket there are 10 different pairs of shoes in a basket. A thief quickly grabs 4 pieces of shoes from the basket at random and runs away. What is the probability that he gets at least 1 complete pair?

Hint. Below we outline two approaches, but only one of them is correct.

- First select one pair, the other two pieces of shoes can be taken arbitrarily, another pair, or any two of the remaining shoes, i.e.:

$$\frac{10\binom{18}{2}}{\binom{20}{4}}$$

- Find the probability of not having a complete pair at all. This can be done by selecting a single shoe, and then putting its matching pair aside. Keep in mind that the order of the selection does not count. Then passing to the complement event, we obtain

$$1 - \frac{\frac{20 \cdot 18 \cdot 16 \cdot 14}{4!}}{\binom{20}{4}}$$

Check out that the two probabilities do not coincide! Which one is correct (if any)?

The first idea is certainly wrong, for the following reason. If we number the pairs, and distinguish the Right and Left shoes, then we can label the shoes in the basket like 1R, 1L, 2R, 2L, and so forth up to 10R, 10L. In the first approach in the numerator we mentioned both the 1R, 1L, 2R, 2L, and the 2R, 2L, 1R, 1L selections, although they are identical samples!

We make a mistake exactly when the sample contains two complete pairs, and all such samples are counted twice. We can fix this by subtracting the number of samples that are duplicated. This way we can also see that the second approach is correct, because:

$$\frac{10\binom{18}{2} - \binom{10}{2}}{\binom{20}{4}} = 1 - \frac{\frac{20 \cdot 18 \cdot 16 \cdot 14}{4!}}{\binom{20}{4}}$$

Verify! Our earlier remark is still valid: we more frequently duplicate some cases, and less frequently overlook and ignore some cases.

Problem 13

From a deck of 52 playing cards we take 5 cards at random without replacement.

(a) Find the probability that all 5 are clubs AND there is at least one Ace.

(b) Find the probability that all 5 are clubs OR there is at least one Ace.

Hint. (a) Consider the following events:

$$A = \{\text{all 5 cards are clubs}\} \quad B = \{\text{there is at least one Ace}\}$$

then

$$P(A \cap B) = \frac{\binom{12}{4}}{\binom{52}{5}}$$

(b) Using the notations above

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{\binom{13}{5}}{\binom{52}{5}} + 1 - \frac{\binom{48}{5}}{\binom{52}{5}} - \frac{\binom{12}{4}}{\binom{52}{5}}$$

where we used the complement event on evaluating $P(B)$.

Problem 14

Alice and Bob make a shooting contest. They shoot alternately, the winner is who first hits the target. Alice is the better shooter, she hits the target with probability 1/2, while Bob only with probability 1/3. Alice generously offers the first shot to Bob. Is this a fair contest?

Hint. The contest is fair if the winning probabilities coincide. Let A denote the event that Alice wins. If Alice wins, then there were an even number of shots. Denote by C_k the event that the game terminates at the k-th shot, $k = 1, 2, 3, \ldots$. Then

$$A = C_2 \cup C_4 \cup C_6 \cup \ldots = \bigcup_{k=1}^{\infty} C_{2k}$$

If C_{2k} occurs, then previously Alice failed $k-1$ times, and Bob failed k times. This means:

$$P(C_{2k}) = \left(\frac{1}{2}\right)^{k-1} \cdot \left(\frac{2}{3}\right)^k \cdot \frac{1}{2} = \left(\frac{1}{3}\right)^k$$

for every $k = 1, 2, \ldots$. It is obviuos that the events C_{2k} are mutually exclusive, therefore

$$P(A) = P\left(\bigcup_{k=1}^{\infty} C_{2k}\right) = \sum_{k=1}^{\infty} P(C_{2k}) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1 - 1/3} - 1 = \frac{1}{2}$$

where we took into account that the term $k = 0$ in the very last geometric series is missing. This means that under the above conditions the contest is fair.

Chapter 15

Problem 15

Consider Problem 9 again. For a student selected at random,

(a) if the student passed Microeconomis, what is the probability that the student passed Math as well?

(b) if the student passed Math, what is the probability that the student did not pass Microeconomics?

(c) if the student passed at least one of the subjects, what is the probability that the student passed Math?

(d) if the student passed at least one of the subjects, what is the probability that the student did not pass Microeconomics?

Hint. Use the notations of Problem 9, then:

(a) $P(A|B) = P(A \cap B)/P(B) = 0.54/0.64 = 0.84375$

(b) $P(\overline{B}|A) = P(\overline{B} \cap A)/P(A) = (P(A) - P(A \cap B))/P(A) = (0.72 - 0.54)/0.72 = 0.25$

(c) $P(A|A \cup B) = P(A)/P(A \cup B) = 0.72/0.82 = 0.878$

(d) $P(\overline{B}|A \cup B) = P(\overline{B} \cap (A \cup B))/P(A \cup B) = P(A \cap \overline{B})/P(A \cup B) = (P(A) - P(A \cap B))/P(A \cup B) = (0.72 - 0.54)/0.82 = 0.219$

where in (c) we used the identity $A \cap (A \cup B) = A$, and in (d) we used the identity $\overline{B} \cap (A \cup B) = A \cap \overline{B}$.

Problem 16

Determine the conditional probability $P(A|B)$ if

(a) B implies A.

(b) A implies B.

(c) A and B are exclusive.

(d) A and B are independent.

Hint. Based on the definition of conditional probability (and common sense), we get the following.

(a) In that case $B \subset A$, hence $A \cap B = B$, and consequently $P(A|B) = P(A \cap B)/P(B) = P(B)/P(B) = 1$.

(b) In that case $A \subset B$, hence $A \cap B = A$, and consequently $P(A|B) = P(A \cap B)/P(B) = P(A)/P(B)$.

(c) In that case $A \cap B = \emptyset$, hence $P(A|B) = P(A \cap B)/P(B) = P(\emptyset)/P(B) = 0$.

(d) In that case $P(A \cap B) = P(A) \cdot P(B)$, hence $P(A|B) = P(A \cap B)/P(B) = P(A) \cdot P(B)/P(B) = P(A)$.

Problem 17

Alice and Bob now play the following game. They toss a die, and then flip a pair of coins (simultaneously) as many times as shown on the die. If at least once Head–Head (HH) occurs, then Alice wins, in the opposite situation Bob wins. Is this a fair game?

Hint. The game is fair if the winning probabilities coincide. Let B denote the event that Bob wins, and denote by C_k the event that the outcome on the die is k (where $k = 1, 2, \ldots, 6$). We can easily check that the events $C_1, C_2, \ldots, C_6$ form a partition of the sample space. Making use of the Theorem of Total Probability we obtain

$$\begin{aligned} P(B) &= \sum_{k=1}^{6} P(B|C_k)P(C_k) = \sum_{k=1}^{6} \left(\frac{3}{4}\right)^k \cdot \frac{1}{6} \\ &= \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1 - (\frac{3}{4})^6}{1 - \frac{3}{4}} = \frac{1}{2} \cdot \left(1 - \left(\frac{3}{4}\right)^6\right) < \frac{1}{2} \end{aligned}$$

Thus the game is not fair. By a slight margin it is advantageous to Alice.

Problem 18

From a deck of 52 playing cards, 2 cards have been lost (unknown). We take a sample of 4 cards at random without replacement.

(a) Find the probability that there are exactly 2 Aces in the sample.

(b) If there are exactly 2 Aces in the sample, what is the probability that both of the lost cards are Aces?

Hint. Let B_0, B_1 and B_2 denote the events that no Ace, one Ace and 2 Aces are lost, respectively. They form a partition of the sample space.

Denote by A the event that the sample contains exactly 2 Aces. Then: (a) Use the theorem of Total probability:

$$\begin{aligned} P(A) &= P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= \frac{\binom{46}{2}\binom{4}{2}}{\binom{50}{4}} \cdot \frac{\binom{48}{2}}{\binom{52}{2}} + \frac{\binom{47}{2}\binom{3}{2}}{\binom{50}{4}} \cdot \frac{\binom{48}{1}\cdot\binom{4}{1}}{\binom{52}{2}} + \frac{\binom{48}{2}}{\binom{50}{4}} \cdot \frac{\binom{4}{2}}{\binom{52}{2}} \end{aligned}$$

(b) Use the Bayes' Rule:

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2)}$$

Problem 19

From a deck of 52 playing cards, 2 cards have been lost (we do not know which). We take a sample of 4 cards at random without replacement.

(a) Find the probability that at least 1 King is taken.

(b) If at least one of the 4 cards is a King, what is the probability that both of the lost cards are Kings?

Hint. Let B_0, B_1 and B_2 denote the events that no King, one King and 2 Kings are lost, respectively. They form a partition of the sample space.

Denote by A the event that the sample contains at least one King. Then:

(a) Use the Theorem of Total Probability (ATTENTION: it is reasonable to use the complement of A):

$$\begin{aligned} P(A) &= P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= \left(1 - \frac{\binom{46}{4}}{\binom{50}{4}}\right) \cdot \frac{\binom{48}{2}}{\binom{52}{2}} + \left(1 - \frac{\binom{47}{4}}{\binom{50}{4}}\right) \cdot \frac{\binom{48}{1}\cdot\binom{4}{1}}{\binom{52}{2}} + \left(1 - \frac{\binom{48}{4}}{\binom{50}{4}}\right) \cdot \frac{\binom{4}{2}}{\binom{52}{2}} \end{aligned}$$

(b) Use the Bayes' Rule:

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2)}$$

Problem 20

In a country the ratio of infected people in an epidemics is 6%. The probability that a new medical device diagnoses a sick person correctly is 0.96, while the probability that it diagnoses someone incorrectly as having the disease is 0.05.

(a) Find the probability that the device diagnoses a person as having the disease.

(b) If the device diagnoses a person as having the disease, what is the probability that the person is really infected?

Hint. For a person selected at random, set

$$B_1 = \{\text{sick}\} \quad B_2 = \{\text{not sick}\} \quad A = \{\text{the device diagnoses as sick}\}$$

Then the events B_1 and B_2 form a partition of the sample space.

(a) Based on the Theorem of total probability:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = 0.96 \cdot 0.06 + 0.05 \cdot 0.94$$

(b) Based on the Bayes' Rule:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} = \frac{0.96 \cdot 0.06}{0.96 \cdot 0.06 + 0.05 \cdot 0.94}$$

Chapters 16 and 17

Problem 21
The density of a random variable X is defined by

$$f(x) = \begin{cases} \frac{A}{(1-x)^2} & \text{if } x > 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of the parameter A!
(b) Find the median of X, i.e. a real number a so that $P(X \geq a) = 1/2$.
(c) Does X have a mean?

Hint. (a)

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_2^{\infty} \frac{A}{(1-x)^2}\,dx = \left[\frac{A}{1-x}\right]_2^{\infty} = A$$

and therefore, $A = 1$.
(b)

$$1/2 = P(X \geq a) = \int_a^{\infty} \frac{1}{(1-x)^2}\,dx = \left[\frac{1}{1-x}\right]_a^{\infty} = \frac{1}{a-1}$$

and it follows that $a = 3$. Recall that X is continuously distributed, and hence

$$P(X < 3) = P(X > 3) = \frac{1}{2}$$

(c) The mean does not exist, since the improper integral

$$\int_2^{\infty} \frac{x}{(1-x)^2}\,dx = \int_2^{\infty} \frac{1}{x-1}\,dx + \int_2^{\infty} \frac{1}{(1-x)^2}\,dx$$

is divergent. Indeed, the first improper integral on the right-hand side is infinite.

Problem 22
The density function of the random variable X is defined by

$$f(x) = \begin{cases} Ae^{-3x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the unknown parameter A.
(b) Evaluate the conditional probability $P(X > 9|X > 6)$.
(c) Which one is bigger, the median of X, or the mean of X?

Hint. (a) Finding the parameter A:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = A\int_0^{\infty} e^{-3x}\,dx = A\left[-\frac{e^{-3x}}{3}\right]_0^{\infty} = \frac{A}{3}$$

this implies $A = 3$.
(b) Take into account that the event $\{X > 9\}$ implies the event $\{X > 6\}$, so by the dfinition of the conditional probability we get

$$P(X > 9|X > 6) = \frac{P(X > 9)}{P(X > 6)} = \frac{\int_9^{\infty} 3e^{-3x}\,dx}{\int_6^{\infty} 3e^{-3x}\,dx} = \frac{e^{-27}}{e^{-18}} = e^{-9} = P(X > 3)$$

The result is interesting, it seems as if the conditional probability depended on the difference only.
(c) Referring to Example 17.12 of Chapter 17 in the online lectures, the mean of X is

$$E(X) = \frac{1}{3}$$

since here $\lambda = 3$. On the other hand, for the unknown value a of the median we get

$$\frac{1}{2} = P(X > a) = \int_a^{\infty} 3e^{-3x}\,dx = \left[-e^{-3x}\right]_a^{\infty} = e^{-3a}$$

From this equation we conclude

$$a = \frac{1}{3} \ln 2$$

which is less than the expected value, since $\ln 2 < 1$. In view of the continuous distribution, we obtain again:

$$P(X < a) = P(X > a) = \frac{1}{2}$$

Problem 23

The density of a random variable X is given by

$$f(x) = \begin{cases} \frac{2A}{x^3} & \text{if } x > 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of the parameter A!

(b) Evaluate the mean of X.

(c) Find the variance of X.

Hint. (a)

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{\infty} \frac{2A}{x^3}\,dx = \left[-\frac{A}{x^2}\right]_2^{\infty} = \frac{A}{4}$$

and hence $A = 4$.

(b)

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{2}^{\infty} \frac{8}{x^2}\,dx = \left[-\frac{8}{x}\right]_2^{\infty} = 4$$

(c) Consider the second moment of X:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{2}^{\infty} \frac{4}{x}\,dx = [4 \ln x]_2^{\infty}$$

which is infinite, hence the second moment does not exist. Thus, X has no variance.

Problem 24

Keep tossing a die until we get two successive identical numbers. Find the mean of the number of tosses.

Hint. let X denote the number of tosses. Verify that the distribution of X is defined by

$$P(X = k) = \left(\frac{5}{6}\right)^{k-2} \cdot \frac{1}{6} \qquad k = 2, 3, \ldots$$

The mean of this variable is given by

$$E(X) = \sum_{k=2}^{\infty} k P(X = k) = \frac{1}{6} \sum_{k=2}^{\infty} k \left(\frac{5}{6}\right)^{k-2}$$

Based on our studies on power series, for each real number $0 < x < 1$ we have

$$\sum_{k=2}^{\infty} k x^{k-2} = \sum_{k=2}^{\infty} (k-1) x^{k-2} + \sum_{k=2}^{\infty} x^{k-2} = \frac{1}{(1-x)^2} + \frac{1}{1-x} = \frac{2-x}{(1-x)^2}$$

By substituting $x = 5/6$, we obtain

$$E(X) = \frac{1}{6} \sum_{k=2}^{\infty} k \left(\frac{5}{6}\right)^{k-2} = \frac{1}{6} \cdot \frac{7/6}{1/36} = 7$$

Chapter 18

Problem 25

An insurance company has 6500 clients. Every client pays an annual fee of 5000 Ft. In a given year the probability that a client files an insurance claim is 0.002 independently from each other. In the case of a claim the company pays 2 million Ft to the client.

(a) Find the probability that the annual profit of the company exceeds 10 million Ft. (Give just the formula, do not evaluate.)

(b) Find the expected annual profit of the company.

Hint. (a) Let X denote the number of claims in the given year. This is a Bernoulli experiment, therefore, X has binomial distribution, with parameters $n = 6500$ and $p = 0.002$.

The total annual revenue of the company is 32 500 000.- Ft. To reach a profit of more than 10 million, the amount paid by the company has to be less than 22 500 000.- Ft. That means the number of claims should be at most 11. Thus

$$P(X \leq 11) = \sum_{k=0}^{11} \binom{6500}{k} 0.002^k \cdot 0.998^{6500-k}$$

(b) Let Y denote the annual profit of the company, then $Y = 32\,500\,000 - 2\,000\,000 \cdot X$. Based on the properties of the mean, we get

$$E(Y) = 32\,500\,000 - 2\,000\,000 \cdot E(X) = 6\,500\,000$$

since for the binomial distribution we have $E(X) = np = 13$.

Problem 26

We toss a pair of playing dice (simultaneously). Let A be the event that the sum is 7. Carry out this experiment 30 times in a row, independently from each other, and let X be the number how many times A occured. Find the distribution of the random variable X and specify its mean and variance!

Hint. As we have seen, the probability of A is

$$P(A) = \frac{1}{6}$$

The procedure described above is a Bernoulli-experiment, hence X is binomially distributed, i.e.

$$P(X = k) = \binom{30}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{30-k}$$

where $k = 0, 1, 2, \ldots, 30$, with parameters $n = 30$ and $p = 1/6$. Consequently

$$E(X) = 30 \cdot \frac{1}{6} = 5 \qquad \text{and} \qquad Var(X) = 30 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{6}$$

Problem 27

A binomially distributed random variable X has a mean of 20 and a standard deviation of 4. Find the probability $P(60 < X < 120)$. (Give just the formula, do not evaluate!)

Hint. For the binomial distribution we have $E(X) = np = 20$, and $D(X) = \sqrt{np(1-p)} = 4$. By solving this system of two equations, we obtain $n = 100$ and $p = 0.2$. Consequently

$$P(60 < X < 120) = P(61 \leq X \leq 100) = \sum_{k=61}^{100} \binom{100}{k} 0.2^k \cdot 0.8^{100-k}$$

since anything above 100 comes with probability 0.

Problem 28

Prove that the geometric distribution is *memoryless*. That means: if X has geometric distribution with a parameter $0 < p < 1$, then for every integer $k \geq 1$ we have

$$P(X = k + 1 | X > k) = P(X = 1) = p$$

(For example, if a playing die is tossed a million times in a row without having 6, then the probability that the very next toss is 6, is still $1/6$.)

Hint. Consider the definition of the conditional probability on the left hand side of the above equality. Keep in mind that the event $\{X = k + 1\}$ implies the event $\{X > k\}$ (subset!).

$$P(X = k+1|X > k) = \frac{P(X = k+1)}{P(X > k)} = \frac{(1-p)^k p}{\sum_{j=k+1}^{\infty}(1-p)^{j-1}p} = \frac{(1-p)^k p}{(1-p)^k \sum_{j=1}^{\infty}(1-p)^{j-1}p} = p$$

since in the very last sum we have

$$\sum_{j=1}^{\infty}(1-p)^{j-1}p = 1$$

by the definition of the geometric distribution. Please verify!

Analogously, we can also verify the next (more general) memoryless property:

$$P(X = k + j | X > k) = P(X = j)$$

for every integer j. Give an interpretation of this formula!

Problem 29

For a random variable X with Poisson-distribution it is given that $P(X < 2) = 4e^{-3}$. Find the mean and the standard deviation of X.

Hint. We need to specify the unknown parameter λ of the Poisson distribution. By our condition

$$P(X < 2) = P(X = 0) + P(X = 1) = e^{-\lambda} + \lambda e^{-\lambda} = (\lambda + 1)e^{-\lambda} = 4e^{-3}$$

From this equation we have $\lambda = 3$. Therefore

$$E(X) = \lambda = 3 \qquad \text{és} \qquad D(X) = \sqrt{\lambda} = \sqrt{3}$$

Problem 30

A communication network consists of n components, any of which functions properly with a given probability $0 < p < 1$, independently from each other. The network is reliable if at least one half of the components function properly.

For what value of p is a 5-component network more reliable than a 3-component network?

Hint. A 5-component network is more reliable than a 3-component network, if it works properly with higher probability. This means that for an unspecified p we must have

$$\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 > \binom{3}{2}p^2(1-p) + p^3$$

ATTENTION: Bernoulli experiment, i.e. the number of functioning components is binomially distributed! By expanding the binomial coefficients and simplifying, we come to the inequality

$$3(1-p)^2(2p-1) > 0$$

whose solution is $p > 1/2$.

Chapter 19

Problem 31

There are 12 bottles of wine in a box. The amount of wine filled in a bottle is normally distributed with a mean of 75 cl and standard deviation of 1 cl. Let A denote the event that there are at most 5 bottles in the box, where the amount of wine is NOT between 73 and 77 cl. Find $P(A)$.

Hint. Let X denote the amount of wine in a bottle. Then X is normally distributed with parameters $m = 75$ and $\sigma = 1$. Thus

$$p = 1 - P(73 < X < 77) = 1 - (\Phi(2) - \Phi(-2)) = 1 - (2\Phi(2) - 1) = 2 - 2\Phi(2)$$

is the probability that the amount of wine is NOT between 73 and 77 in a bottle. Therefore,

$$P(A) = \sum_{k=0}^{5} \binom{12}{k} p^k (1-p)^{12-k}$$

where p is the probability above. (ATTENTION: this is a Bernoulli experiment!)

Problem 32

For a uniformly distributed random variable X it is given that $E(X) = 3/2$ and $D(X) = 7/\sqrt{12}$.

(a) Determine the density function of X!

(b) Find the probability $P(-3 < X < 4)$.

Hint. First, we need to determine the unknown parameters a and b of the uniform distribution.

$$E(X) = \frac{a+b}{2} = \frac{3}{2} \quad \text{and} \quad Var(X) = \frac{(b-a)^2}{12} = \frac{49}{12}$$

from these two equations: $a = -2$ and $b = 5$. Thus (a):

$$f(x) = \begin{cases} \frac{1}{7} & \text{if } -2 < x < 5 \\ 0 & \text{elsewhere} \end{cases}$$

is the desired density function.

(b) $P(-3 < X < 4) = P(-2 < X < 4) = 6/7$, because anything below -2 comes with 0 probability.

Problem 33

(a) On a given day at the Budapest Stock Exchange the closing price of a stock is normally distributed with a mean of $m = 100$ and standard deviation of $\sigma = 4$. If we buy a stock for 100 Ft at opening, what is the probability the we realize a 10% profit at closing time? (Use the Φ function!)

(b) By using the cumulative distribution function Φ of the standard normal distribution, express the probability that the closing price remains within a 5% margin of 100 Ft.

Hint. (a) Let X denote the closing price of the stock, and F is its cumulative distribution function. Then $P(X > 110) = 1 - F(110) = 1 - \Phi(2.5)$.

(b) $P(95 < X < 105) = F(105) - F(95) = \Phi(1.25) - \Phi(-1.25) = 2\Phi(1.25) - 1$.

Problem 34

The life time of certain electronic parts is normally distributed with a mean of $m = 3$ years and a standard deviation of $\sigma = 1.5$ years. If we take 100 pieces of such parts, what is the probability that at least 60 of them have a life time longer than 6 years? (Give just the formula in terms of the Φ function.)

Hint. Let X be the life time of an arbitrarily taken part, and let F denote the cumulative distribution function of X. Then

$$P(X > 6) = 1 - F(6) = 1 - \Phi(2)$$

is the probability that one part has a life time longer than 6 years. The complete answer is:

$$\sum_{k=60}^{100} \binom{100}{k} (1 - \Phi(2))^k \Phi(2)^{100-k}$$

ATTENTION: recognize the Bernoulli experiment!

Problem 35

Evaluate the probability $P(-2 < X < 10)$, if
(a) X is normally distributed with parameters $m = 4$ and $\sigma = 5$ (use the Φ function).
(b) X is uniformly distributed on the interval $[0, 12]$.
(c) X is exponentially distributed with the parameter $\lambda = 1/5$.
(d) X has a Poisson-distribution with parameter $\lambda = 2$.

Hint.
(a) $P(-2 < X < 10) = F(10) - F(-2) = \Phi(1.2) - \Phi(-1.2) = 2\Phi(1.2) - 1$.
(b) $P(-2 < X < 10) = P(0 < X < 10) = 5/6$.
(c) $P(-2 < X < 10) = P(0 < X < 10) = \int_0^{10} \frac{1}{5} e^{-x/5}\, dx = 1 - e^{-2}$.
(d) $P(-2 < X < 10) = P(0 \le X \le 9) = \sum_{k=0}^{9} \frac{2^k}{k!} e^{-2}$.

Problem 36

Let X a random variable with exponential distribution, whose mean is $E(X) = 3$.
(a) Evaluate the expected value $E(X^2 - 2X)$.
(b) Find the conditional probability $P(X > 9|X > 6)$.

Hint. Based on the mean, the parameter of the exponential distribution is $\lambda = 1/3$. Hence
(a) $E(X^2 - 2X) = E(X^2) - 2E(X) = 2/\lambda^2 - 2/\lambda = 18 - 6 = 12$.
(b) In view of the memoryless property of the exponential distribution, we have

$$P(X > 9|X > 6) = P(X > 3) = \int_3^\infty \frac{1}{3} e^{-x/3}\, dx = \left[-e^{-x/3}\right]_3^\infty = \frac{1}{e}$$

Problem 37

The life time of a small device is a random variable X with a mean of $m = 4$ years. Let A denote the event that the life time of the device is between 3 and 5 years.
(a) Find $P(A)$ if X is normally distributed with parameters $m = 4$ and $\sigma = 0.4$. Express your answer in terms of the Φ function.
(b) Find $P(A)$ if X is exponentially distributed.
(c) Find $P(A)$ if X is uniformly distributed on the interval $[2, 6]$.

Hint. (a) Let F be the cumulative distribution function of X, then

$$P(3 < X < 5) = F(5) - F(3) = \Phi(\frac{1}{0.4}) - \Phi(-\frac{1}{0.4}) = \Phi(2.5) - \Phi(-2.5) = 2\Phi(2.5) - 1$$

(b) If X exponentially distributed, then its parameter is $\lambda = 1/4$, and therefore,

$$P(3 < X < 5) = \int_3^5 \frac{1}{4} e^{-x/4}\, dx = \left[-e^{-x/4}\right]_3^5 = e^{-3/4} - e^{-5/4}$$

(c) If X is uniformly distributed, then the probability is proportional to the length of the subinterval, and consequently:

$$P(3 < X < 5) = 2/4 = 1/2$$

Problem 38

TRUE or FALSE:
(a) If A and B are independent events, then so are $\overline{A}$ and $\overline{B}$.
(b) If $P(A) \le P(B)$, then $A \subset B$.
(c) If X is uniformly distributed on the interval $[0, 6]$, then $Var(X) = 3$.

(d) If X has Poisson-distribution with parameter $\lambda > 0$, then $E(X^2) = \lambda^2$.

(e) There exists a random variable X with Poisson-distribution so that $E(X) = 4$ and $E(X^2) = 20$.

(f) For a random variable X with Poisson-distribution we have $E(X) = 49$. Then

$$P(X \le D(X)) = \sum_{k=0}^{7} \frac{49^k}{k!} e^{-49}$$

(g) If for the events A and B we have $P(A) \cdot P(B) \neq 0$ and $P(A|B) = P(B|A)$, then $P(A) = P(B)$.

(h) If $A \subset B$, then $P(A) < P(B)$.

(i) There exists an exponentially distributed random variable X with $E(X) = 4$ and $E(X^2) = 32$.

(j) If A and B are independent, then so are A and $A \cap B$.

(k) If for the events A and B we have $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$, then $P(A|B) = 0.4$.

(l) For a random variable X with Poisson-distribution and parameter $\lambda = 6$ necessarily $E(X^2) = 36$.

Problem 39

A machine discharges beer into bottles. The amount of beer filled in a bottle is a normally distributed random variable with a mean $m = 5$ dl and standard deviation $\sigma = 0.05$ dl (independently). A bottle is defective if the amount of beer in the bottle is not between 4.9 and 5.1 dl.

(a) What is the probability that a bottle is defective? (Use the Φ function!)

(b) Find the probability that at most 5 bottles are defective out of 1000 bottles. (Give just the formula!)

(c) Find the expected number of defective bottles out of 1000 bottles!

Hint. (a) Let X denote the amount of beer in a bottle, and denote by F its cumulative distribution function. Then

$$p = 1 - P(4.9 < X < 5.1) = 1 - (F(5.1) - F(4.9)) = 1 - (\Phi(2) - \Phi(-2)) = 2 - 2\Phi(2)$$

is the probability that a bottle is defective.

(b) Since this is a Bernoulli-experiment, we have:

$$\sum_{k=0}^{5} \binom{1000}{k} p^k (1-p)^{1000-k}$$

(c) If Y denotes the number of defective bottles, then Y is binomially distributed (Bernoulli!), with parameters $n = 1000$ and $p = 2 - 2\Phi(2)$. Thus $E(Y) = 1000(2 - 2\Phi(2))$.

Problem 40

Consider a random variable X with a mean $m = 1$ and standard deviation $\sigma > 0$. Find the probability $P(m - 2\sigma < X < m + 2\sigma)$ if

(a) X is normally distributed (Use the Φ function!),

(b) X is exponentially distributed,

(c) X has Poisson-distribution (give just the formula, do not evaluate!).

Hint. (a) If F is the cumulative distribution function of X, then $P(m - 2\sigma < X < m + 2\sigma) = F(m + 2\sigma) - F(m - 2\sigma) = \Phi(2) - \Phi(-2) = 2\Phi(2) - 1$.

(b) If X is exponentially distributed, then $\lambda = 1$, and $m = \sigma = 1$, thus

$$P(-1 < X < 3) = P(0 < X < 3) = \int_0^3 e^{-x}\, dx = \left[-e^{-x}\right]_0^3 = 1 - e^{-3}$$

(c) If X has Poisson-distribution, then $\lambda = 1$ and $\sigma = \sqrt{\lambda} = 1$, therefore

$$P(-1 < X < 3) = P(0 \le X \le 2) = \sum_{k=0}^{2} \frac{1}{k!} e^{-1}$$

Chapter 20

Problem 41

The joint distribution of the discrete random variables X and Y is given by the chart below.

$Y \setminus X$	0	1	2	3
0	0.15	0.08	0.13	0.04
1	0.04	0.2	0.08	0
2	0.03	0	0.05	0.2

(a) Find the marginal distribution of the variable Y.
(b) Are X and Y independent?
(c) Find the conditional probability $P(X > 1|Y < 2)$.
(d) Evaluate the variance $Var(Y)$.

Hint. (a) The marginal distributions are given in the following table.

$Y \setminus X$	0	1	2	3	
0	0.15	0.08	0.13	0.04	0.4
1	0.04	0.2	0.08	0	0.32
2	0.03	0	0.05	0.2	0.28
	0.22	0.28	0.26	0.24	

(b) Not independent, since $0.15 = P(X = 0, Y = 0) \neq P(X = 0) \cdot P(Y = 0) = 0.22 \cdot 0.4 = 0.088$.
(c) By the definition of the conditional probability:

$$P(X > 1|Y < 2) = \frac{P(X > 1, Y < 2)}{P(Y < 2)} = \frac{0.25}{0.72}$$

(d) Clearly $E(Y) = 0 \cdot 0.4 + 1 \cdot 0.32 + 2 \cdot 0.28 = 0.88$, and the second moment is given by $E(Y^2) = 0 \cdot 0.4 + 1 \cdot 0.32 + 4 \cdot 0.28 = 1.44$. Hence, $Var(Y) = 1.44 - 0.88^2$.

Problem 42

We toss a die twice in a row. Let X be the absolute value of the difference of the two numbers, and let Y denote how many even numbers we get.
(a) Find the joint distribution of X and Y.
(b) Are X and Y independent?
(c) Determine the conditional probability $P(Y > 1|X > 3)$.

Hint. (a) The joint distribution is given by this table:

$Y \setminus X$	0	1	2	3	4	5
0	3/36	0	4/36	0	2/36	0
1	0	10/36	0	6/36	0	2/36
2	3/36	0	4/36	0	2/36	0

(b) Not independent, for instance $3/36 = P(X = 0, Y = 0) \neq P(X = 0) \cdot P(Y = 0) = 6/36 \cdot 9/36$.
(c) By the definition of the conditional probability:

$$P(Y > 1|X > 3) = \frac{P(X > 3, Y > 1)}{P(X > 3)} = \frac{2/36}{6/36} = \frac{1}{3}$$

Problem 43

The joint density function of X and Y is defined by

$$f(x,y) = \begin{cases} a(x+y^2) & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of the parameter a.
(b) Evaluate the mean $E(Y)$.
(c) Are X and Y independent?

Hint. (a) Integrating first with respect to y and then with respect to x, we get:

$$1 = a\int_0^1\int_0^1 (x+y^2)\,dy\,dx = a\int_0^1 \left[xy+y^3/3\right]_0^1 dx = a\int_0^1 (x+1/3)\,dx = a(1/2+1/3)$$

and this implies $a = 6/5$.
(b) Find the marginal density of Y. For a fixed $0<y<1$ we have:

$$f_Y(y) = \frac{6}{5}\int_0^1 (x+y^2)\,dx = \frac{6}{5}\left[\frac{x^2}{2}+xy^2\right]_0^1 = \frac{6}{5}\left(\frac{1}{2}+y^2\right)$$

Therefore, the marginal density of Y is

$$f_Y(y) = \begin{cases} \frac{6}{5}\left(\frac{1}{2}+y^2\right) & \text{if } 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Finally, compute the mean:

$$E(Y) = \int_{-\infty}^{\infty} y\cdot f_Y(y)\,dy = \frac{6}{5}\int_0^1 \left(\frac{y}{2}+y^3\right)\,dy = \frac{6}{5}\left[\frac{y^2}{4}+\frac{y^4}{4}\right]_0^1 = \frac{3}{5}.$$

(c) Not independent, since $f(x,y) \neq f_X(x)\cdot f_Y(y)$.

Problem 44

The joint density function of the random variables X and Y is given by:

$$f(x,y) = \begin{cases} Axy, & \text{if } 0<x<1,\ 0<y<2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the unknown parameter A.
(b) Are X and Y independent?
(c) Find the expected value $E(X)$.

Hint. (a) Finding the unknown parameter A goes like this:

$$1 = A\int_0^1\int_0^2 xy\,dy\,dx = A\int_0^1 \left[\frac{xy^2}{2}\right]_0^2 dx = A\int_0^1 2x\,dx = A\left[x^2\right]_0^1 = A$$

(b) By integrating we see that the two marginal densities are:

$$f_X(x) = \begin{cases} 2x & \text{if } 0<x<1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{és} \quad f_Y(y) = \begin{cases} \frac{y}{2} & \text{if } 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, $f(x,y) = f_X(x)\cdot f_Y(y)$ for every x and y, this means that X and Y are independent.
(c) $E(X) = \int_0^1 2x^2\,dx = 2/3$.

Problem 45

The joint density function of the random variables X and Y is:

$$f(x,y) = \begin{cases} x^2 + x + Ay & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the unknown parameter A.

(b) Find the marginal density functions.

(d) Evaluate the probability $P(X + Y < 1)$.

Hint. (a) Finding the unknown parameter A:

$$1 = \int_0^1 \int_0^1 (x^2 + x + Ay)\, dy\, dx = \int_0^1 \left[x^2 y + xy + A\frac{y^2}{2} \right]_0^1 dx = \int_0^1 \left(x^2 + x + \frac{A}{2} \right) dx = \frac{1}{3} + \frac{1}{2} + \frac{A}{2}$$

and this implies $A = 1/3$.

(b) By examining the previous integral in (a), we see

$$f_X(x) = \begin{cases} x^2 + x + \frac{1}{6} & \text{if } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{3} + \frac{1}{2} + \frac{y}{3} & \text{if } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(c) We integrate on the isosceles right triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$

$$\begin{aligned} P(X + Y < 1) &= \int_0^1 \int_0^{1-x} \left(x^2 + x + \frac{y}{3} \right) dy\, dx = \int_0^1 \left[x^2 y + xy + \frac{y^2}{6} \right]_0^{1-x} dx \\ &= \int_0^1 \left(x^2(1-x) + x(1-x) + \frac{1}{6}(1-x)^2 \right) dx = \frac{1}{2} - \frac{1}{4} - \frac{1}{18} \end{aligned}$$

ATTENTION: checking Figure 1 in the file Abrak.pdf is helpful!

Problem 46

The joint density of the random variables X and Y is defined by

$$f(x,y) = \begin{cases} 6x & \text{if } 0 < x < 1,\ 0 < y < 1,\ x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the probability $P(Y < 1/2)$.

(b) Evaluate the conditional probability $P(X < 1/2 | Y < 1/2)$.

Hint. (a) Take a look at Figure 2 of the file Abrak.pdf

$$P(Y < 1/2) = \int_0^{1/2} \int_0^{1-y} 6x\, dx\, dy = \int_0^{1/2} [3x^2]_0^{1-y}\, dy = \int_0^{1/2} 3(1-y)^2\, dy = 7/8$$

(b) Using Figure 2 again, we can see

$$P(X < 1/2, Y < 1/2) = \int_0^{1/2} \int_0^{1/2} 6x\, dx\, dy = \int_0^{1/2} [3x^2]_0^{1/2} = 3/8$$

Thus, by the definition of the conditional probability:

$$P(X < 1/2 | Y < 1/2) = \frac{P(X < 1/2, Y < 1/2)}{P(Y < 1/2)} = \frac{3/8}{7/8} = \frac{3}{7}$$

Chapter 21

Problem 47

The joint density of the random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y,\ 0 < y < 1, \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the probability $P(X + Y > 1)$.
(b) Evaluate the conditional probability $P(X < 1/2 | Y < 1/2)$.
(c) Find the covariance of X and Y.

Hint. (a) In view of Figure 3 (see: Abrak.pdf, look at the blue dotted line):

$$\begin{aligned} P(X+Y>1) &= \int_{1/2}^{1}\int_{1-y}^{y}\frac{1}{y}\,dx\,dy = \int_{1/2}^{1}\left[\frac{x}{y}\right]_{1-y}^{y}dy = \int_{1/2}^{1}\left(1-\frac{1-y}{y}\right)dy \\ &= \int_{1/2}^{1}\left(2-\frac{1}{y}\right)dy = [2y-\ln y]_{1/2}^{1} = 1-\ln 2 \end{aligned}$$

(b) In view of Figure 4:

$$P(X<1/2, Y<1/2) = \int_0^{1/2}\int_0^{y}\frac{1}{y}\,dx\,dy = \int_0^{1/2}\left[\frac{x}{y}\right]_0^{y}dy = \int_0^{1/2}dy = \frac{1}{2}$$

We also see on the picture that $P(Y < 1/2) = P(X < 1/2, Y < 1/2) = 1/2$, and consequently

$$P(X<1/2|Y<1/2) = \frac{P(X<1/2, Y<1/2)}{P(Y<1/2)} = 1$$

This result is not surprising, since on the entire triangle (where the density is not zero) we have that $Y > X$. Therefore, the event $\{Y < 1/2\}$ implies the event $\{X < 1/2\}$.
(c) The mean of X is obtained by integration by parts:

$$\begin{aligned} E(X) &= \int_0^1 x\cdot\int_x^1\frac{1}{y}\,dy\,dx = \int_0^1 x\cdot[\ln y]_x^1\,dx = -\int_0^1 x\ln x\,dx \\ &= -\left[\frac{x^2}{2}\ln x\right]_0^1 + \int_0^1\frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4} \end{aligned}$$

where we exploited the L'Hôpital-Rule. The mean of Y is computed like this:

$$E(Y) = \int_0^1 y\cdot\int_0^y\frac{1}{y}\,dx\,dy = \int_0^1 y\left[\frac{x}{y}\right]_0^y dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

Furthermore, the mean of the product XY is:

$$E(XY) = \int_0^1\int_0^y xy\frac{1}{y}\,dx\,dy = \int_0^1\left[\frac{x^2}{2}\right]_0^y dy = \int_0^1\frac{y^2}{2}\,dy = \frac{1}{6}$$

Finally, we evaluate the covariance:

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

Problem 48

The joint density of the random variables X and Y is defined by

$$f(x,y) = \begin{cases} 6e^{-2x-3y} & \text{if } x > 0, \text{ és } y > 0, \\ 0 & \text{elsewhere} \end{cases}$$

Find the correlation coefficient of X and Y.

Hint. Observe that the joint density is the product of the two densities:

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{and} \qquad f_Y(x) = \begin{cases} 3e^{-3y} & \text{if } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

that are exponential distributions with parameters $\lambda = 2$ and $\mu = 3$ respectively.
ATTENTION: Verify this observation directly by integration!
Since $f(x, y) = f_X(x) \cdot f_Y(y)$ at every point, X and Y are independent, and hence $Corr(X, Y) = 0$.

Problem 49
Let X and Y be independent, exponentially distributed random variables with parameters $\lambda = 1/2$ and $\mu = 1/3$ respectively. Find the expected value

$$E((X+Y)^2)$$

Hint. By the properties of the exponential distribution we have $E(X) = 2$, $E(X^2) = 8$, $E(Y) = 3$ and $E(Y^2) = 18$. On the other hand, making use of the independence

$$E((X+Y)^2) = E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + 2E(X) \cdot E(Y) + E(Y^2)$$

This tells us that $E((X+Y)^2) = 8 + 2 \cdot 2 \cdot 3 + 18 = 38$.

Problem 50
Let X and Y be independent random variables with Poisson-distribution and with parameters $\lambda = 2$ and $\mu = 3$ respectively. Find the expected value

$$E((X-Y)^2)$$

Hint. By the properties of the Poisson-distribution we have $E(X) = 2$, $E(X^2) = Var(X) + E(X)^2 = 6$, and analogously $E(Y) = 3$, $E(Y^2) = Var(Y) + E(Y)^2 = 12$. On the other hand, in view of the independence:

$$E((X-Y)^2) = E(X^2) - 2E(XY) + E(Y^2) = E(X^2) - 2E(X) \cdot E(Y) + E(Y^2)$$

This implies $E((X-Y)^2) = 6 - 2 \cdot 2 \cdot 3 + 12 = 6$.

Problem 51
Let X and Y be independent, normally distributed random variables with parameters $m_1 = 2$, $\sigma_1 = 1$ and $m_2 = 3$, $\sigma_2 = 2$ respectively.
(a) Find $Var(X+Y)$.
(b) Evaluate the mean $E(X^2 - XY + Y^2)$.

Hint. (a) From the parameters we get $Var(X) = 1$ and $Var(Y) = 4$, and by the independence:

$$Var(X+Y) = Var(X) + Var(Y) = 5$$

(b) On the one hand $E(X^2) = Var(X) + E(X)^2 = 5$ and $E(Y^2) = Var(Y) + E(Y)^2 = 13$. On the other hand, from the independence we conclude:

$$E((X^2 - XY + Y^2) = E(X^2) - E(XY) + E(Y^2) = E(X^2) - E(X) \cdot E(Y) + E(Y^2) = 5 - 6 + 13 = 12$$

Problem 52
TRUE or FALSE?
(a) If $Corr(X, Y) = 1/4$, then $Corr(-3X, 5Y) = -1/4$.
(b) If X éand Y are independent, then $D(X+Y) = D(X) + D(Y)$.
(c) If $Corr(X, Y) = 0$, then $E(XY) = E(X) \cdot E(Y)$.
(d) If $Corr(X, Y) = 0$, then X and Y are independent.
(e) If $Z_1, \ldots, Z_n$ are independent random variables with standard normal distribution, then $D(Z_1 + \ldots + Z_n) = n$.

Chapter 22

Problem 53

Let X and Y be independent random variables that are uniformly distributed on the interval $[0, 1]$. Find the density function of the variable $X + Y$.

First Solution. First compute the cumulative distribution function of $X+Y$. Let H denote this function that is $H(x) = P(X+Y < x)$ for all $x \in \mathbb{R}$. Clearly, for every $x \leq 0$ we have $H(x) = P(X+Y < x) = 0$. If we pick a point $0 < x \leq 1$, then (see Figure 5 in Abrak.pdf):

$$H(x) = P(X+Y < x) = \frac{x^2}{2}$$

and if $1 < x \leq 2$, then (see Figure 6 in Abrak.pdf):

$$H(x) = P(X+Y < x) = 1 - \frac{1}{2}(2-x)^2 = 1 - \left(2 - 2x + \frac{x^2}{2}\right) = -1 + 2x - \frac{x^2}{2}$$

Obviously, if $x > 2$, then $H(x) = P(X+Y < x) = 1$. Putting all cases together, we have

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x^2}{2} & \text{if } 0 < x \leq 1 \\ -1 + 2x - \frac{x^2}{2} & \text{if } 1 < x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

The density function h is the derivative of H, i.e.

$$h(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 - x & \text{ha } 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Second Solution. Now we derive the density function h directly from the convolution integral. Both X and Y are uniformly distributed on the interval $[0, 1]$, and denote by f their common density function. Then for every $x \in \mathbb{R}$ the convolution integral possesses the form

$$h(x) = \int_{-\infty}^{\infty} f(t) f(x-t)\, dt$$

Let us see for what values of x is the integrand not zero (whenever it is not zero, it is equal to 1). The necessary and sufficient condition is $0 < t < 1$ és $0 < x - t < 1$. The latter inequality can be rewritten like $t < x < 1 + t$. So, if $0 < x < 1$, then $0 < t < x$, and hence

$$h(x) = \int_0^x 1\, dt = x$$

If $1 < x < 2$, then $x - 1 < t < 1$, and hence

$$h(x) = \int_{x-1}^{1} 1\, dt = [t]_{x-1}^{1} = 1 - (x-1) = 2 - x$$

We conclude that the density function of $X + Y$ is

$$h(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 - x & \text{ha } 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

precisely as we have seen in the first solution.

Problem 54

Let X and Y be independent, binomially distributed random variables with parameters n, p and m, p respectively. Find the distribution of $X + Y$.

Hint. Pick an integer k. In view of the independence

$$\begin{aligned}
P(X+Y=k) &= \sum_{j=0}^{k} P(X=j, Y=k-j) = \sum_{j=0}^{k} P(X=j)\cdot P(Y=k-j) = \\
&= \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j} \cdot \binom{m}{k-j} p^{k-j}(1-p)^{m-k+j} \\
&= p^k(1-p)^{n+m-k} \sum_{j=0}^{k} \binom{n}{j}\cdot\binom{m}{k-j} = \binom{n+m}{k} p^k (1-p)^{n+m-k}
\end{aligned}$$

In the last line we applied the "apple–pear" identity!

Summing up, we have come to a binomial distribution with parameters $n+m$ and p. Let us point out that this argument would not have worked, if the p parameters of X and Y had been different!

Problem 55

In a multiple choice exam there are 10 problems, each one with 5 possible answers, so that exactly one of them correct. At least 4 correct answers are needed to pass to exam. Suppose that a student fills in the exam form completely at random. Find the probability that the student will pass the exam.

Hint. Let X_k denote the score of the student on Problem k that is 1 or 0 depending on whether it is correct or not ($k=1,\ldots,10$). The total score is $X = X_1+\ldots+X_{10}$. The event that the student passes the exam is $\{X \geq 4\}$. The exact value of the probability of this event is

$$P(X \geq 4) = \sum_{k=4}^{10} \binom{10}{k} 0.2^k 0.8^{10-k}$$

since in every Problem the probability of the correct answer is 0.2.

Clearly, X is binomially distributed (Bernoulli-experiment!) with parameters $n=10$ and $p=0.2$, therefore, its mean is $m=2$, and the standard deviation is $\sigma = \sqrt{10\cdot 0.4} \approx 1.26$.

If we now use the approximation by the standard normal distribution (as described in the Central Limit Theorem), we get

$$\begin{aligned}
P(4 \leq X \leq 10) &= P\left(\frac{2}{1.26} \leq \frac{X-2}{1.26} \leq \frac{8}{1.26}\right) \approx P(1.5873 < Z < 6.3492) \\
&= \Phi(6.3492) - \Phi(1.5873) \approx 1 - 0.9441 = 0.0559
\end{aligned}$$

and these are exact values up to 4 decimals. The values of Φ can be found in the Appendix of Textbook-2, or in any spreadsheet program (for instance MS Excel).

CONCLUSION: with a random selection we do not even have a chance of 6% to pass the exam.

Problem 56

In an Economics class the grades of the Probability course are determined by "Grading on the curve". This means that there are no fixed limits, grades depend on the performance of the class. For example the lowest 10% will fail, the highest 10% will get a grade 5, and so forth. If the scores of the class are approximately normally distributed with parameters $m=74$ and $\sigma=8.2$, what is the lowest possible score to pass the exam?

Hint. Let X denote the score of an exam. We look for a real number a so that $P(X<a) \geq 0.1$, and a is the smallest possible number with this property. By converting to the standard normal distribution, we have

$$P(X<a) = P\left(Z < \frac{a-m}{\sigma}\right) = P\left(Z < \frac{a-74}{8.2}\right) \geq 0.1$$

If we look up 0.1 in the table, the closest value to 0.9 is $\Phi(1.29) = 0.9015$, and the symmetry yields $\Phi(-1.29) = 0.0985 < 0.1$. Consequently, if

$$\frac{a-74}{8.2} = -1.29$$

then we obtain $a \approx 63.422$. This means that a student with a score of 64 certainly can pass the exam. A score of 63 is not certainly sufficient!

Problem 57

On the average a new Toyota runs 6 years without major repair. Suppose that the time until the first repair is approximately normally distributed with parameters $m = 6$ years and $\sigma = 1.2$ years. The dealer repairs all cars free while under guarantee. If the dealer is willing to repair 2% of the cars freely, how long a guarantee should he offer?

Hint. For a new Toyota let X denote the length of the time interval until the first repair. We look for a real number a (length of guarantee period), so that $P(X < a) \leq 0.02$, and it is the smallest number with this property (budget constraint).

Converting to the standard normal distribution, we get

$$P(X < a) = P\left(Z < \frac{a-m}{\sigma}\right) = \Phi\left(\frac{a-6}{1.2}\right) \leq 0.02$$

In the table we look up $\Phi(2.06) = 0.9803$, and by the symmetry property $\Phi(-2.06) = 0.0197 < 0.02$. If we set

$$\frac{a-6}{1.2} = -2.06$$

then we have $a \approx 3.528$. Therefore, the dealer can safely offer 3 years of guarantee (even 3.5 years!).

Problem 58

In a given year the number of submitted claims to an insurance company is a random variable with Poisson-distribution with parameter $\lambda > 0$. The probability that a claim is refused by the company is $0 < p < 1$ independently from each other. Find the distribution and the expected value of the number of refused claims.

Hint. Let X denote the number of refused claims, and let Y denote the number of all incoming claims. Clearly, the events $\{Y = n\}$ for $n = 0, 1, 2, \ldots$ form a partition of the sample space.

Pick an integer k arbitrarily. To find the distribution of X we need to determine the probabilities $P(X = k)$. In view of the Theorem of Total Probability we get

$$P(X = k) = \sum_{n=0}^{\infty} P(X = k|Y = n) \cdot P(Y = n)$$

To evaluate this sum, keep in mind that for a fixed integer n the probability $P(X = k|Y = n)$ refers to a Bernoulli-experiment. On the other hand, for $k > n$ we have $P(X = k|Y = n) = 0$ (number of refused claims cannot be bigger than the number of submitted claims). Therefore,

$$\begin{aligned} P(X = k) &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} = \sum_{n=k}^{\infty} \frac{(\lambda p)^k}{k!} \cdot \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} e^{-\lambda} \\ &= \frac{(\lambda p)^k}{k!} e^{-\lambda p} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} e^{-\lambda(1-p)} = \frac{(\lambda p)^k}{k!} e^{-\lambda p} \end{aligned}$$

where we observe that the sum in the last line is 1, since it is the sum of the Poisson distribution with parameter $\lambda(1-p)$.

Consequently, the number of refused claims has Poisson distribution with parameter λp, and hence $E(X) = \lambda p$.

Problem 59

Let Z_1, Z_2 and Z_3 independent, standard normally distributed random variables, and denote by $X = Z_1 + Z_2 + Z_3$ their sum. Express the probability $P(-3 < X < 3)$ in terms of the Φ function.

Hint. As a consequence of independence X is normally distributed with a mean $m = 0$ and standard deviation $\sigma = \sqrt{3}$. Let F denote the cumulative distribution function of X. Then:

$$P(-3 < X < 3) = F(3) - F(-3) = \Phi\left(\frac{3}{\sqrt{3}}\right) - \Phi\left(\frac{-3}{\sqrt{3}}\right) = 2\Phi(\sqrt{3}) - 1$$

in view of the symmetry of Φ.

Chapter 23

Problem 60

Consider a random variable X with a mean of $m = 10$ and standard deviation $\sigma = 3$. By using Chebyshev's Theorem give an estimate on the following probabilities:
(a) $P(5 < X < 15)$
(b) $P(|X - 10| \geq 4)$
(c) $P(3 < X < 16)$
(d) Find the lowest $\varepsilon > 0$ so that $P(|X - 10| < \varepsilon) \geq 0.8$.

Hint. Chebyshev's Theorem states that for every $\varepsilon > 0$ we have

$$P(m - \varepsilon < X < m + \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

(a) Now $\varepsilon = 5$, and hance $P(5 < X < 15) \geq 1 - 9/25 = 16/25$
(b) $P(|X - 10| \geq 4) = 1 - P(6 < X < 14) \leq 1 - (1 - 9/16) = 9/16$ (opposite estimate!)
(c) The event $\{3 < X < 16\}$ is not symmetric about $m = 10$-re. Make it symmetric! Chebyshev's Theorem provides a lower estimate, so the symmetry can be reached by shrinking the event:

$$P(3 < X < 16) \geq P(4 < X < 16) \geq 1 - \frac{9}{36} = \frac{3}{4}$$

(d) The smallest value of ε can be gained from Chebyshev's Theorem:

$$P(|X - 10| < \varepsilon) = P(10 - \varepsilon < X < 10 + \varepsilon) \geq 1 - \frac{9}{\varepsilon^2} = 0.8$$

This means $\varepsilon^2 = 45$, and hence $\varepsilon \approx 6.7$

Problem 61

Give a lower estimate on the probability $P(m - 2\sigma < X < m + 2\sigma)$, and find the exact value under the following conditions:
(a) X is normally distributed,
(b) X is uniformly distributed on the interval $[0, 1]$,
(c) X is exponentially distributed with parameter $\lambda = 1/4$,
(d) the density function of X is defined by:

$$f(x) = \begin{cases} 4x + 2 & \text{ha } -1/2 < x < 0 \\ 2 - 4x & \text{ha } 0 < x < 1/2 \end{cases}$$

Hint. The lower estimate:

$$P(m - 2\sigma < X < m + 2\sigma) \geq 1 - \frac{1}{4} = 0.75$$

(a) In the case of normal distribution we have: $P(m - 2\sigma < X < m + 2\sigma) = 2\Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 0.9544$
(b) In the case of uniform distribution we have $m = 1/2$ and $\sigma = 1/2\sqrt{3}$, therefore

$$P(m - 2\sigma < X < m + 2\sigma) = P\left(\frac{1}{2} - \frac{1}{\sqrt{3}} < X < \frac{1}{2} + \frac{1}{\sqrt{3}}\right) = P(0 < X < 1) = 1$$

(c) In the case of exponential distribution we have $m = 1/\lambda = 4$, and $\sigma = 1/\lambda = 4$, thus

$$\begin{aligned} P(m - 2\sigma < X < m + 2\sigma) &= P(-4 < X < 12) = P(0 < X < 12) = \int_0^{12} \frac{1}{4} e^{-x/4}\, dx = \left[-e^{-x/4}\right]_0^{12} \\ &= 1 - e^{-3} \approx 0.95 \end{aligned}$$

(d) By graphing the function f we see that it is symmetric about the y-axis (even function), so $m = E(X) = 0$. On the other hand, the second moment can be computed this way:

$$E(X^2) = \int_{-1/2}^{1/2} x^2 f(x)\,dx = 2\int_0^{1/2} x^2(2-4x)\,dx = 2\left[\frac{2x^3}{3} - x^4\right]_0^{1/2} = 2\left(\frac{1}{12} - \frac{1}{16}\right) = \frac{1}{24}$$

We conclude that $\sigma = 1/2\sqrt{6} \approx 0.2$. In view of the graph of f we get

$$P(m - 2\sigma < X < m + 2\sigma) \approx \int_{-0.4}^{0.4} f(x)\,dx = 1 - 0.04 = 0.96$$

Conclusion: Chebyshev's Theorem is true for any random variable (if m and σ exist), therefore we cannot expect a high accuracy. The more "widespread" the distribution is, the more precise estimate can be expected.

Problem 62

(a) A multinational company announces openings for 7 executive positions. Applicants are expected to have excellent mathematical abilities (primarily in Probability Theory). For the 100 applicants the company gives a test in mathematics. The average score of the applicants is $m = 60$ points with a standard deviation of $\sigma = 6$. We have no information about the distribution of the scores. If I applied for one of the positions, and my score is 83, can I count on getting the job?

(b) What can we say if the scores are approximately normally distributed?

(c) What can we say if the scores are approximately uniformly distributed?

Hint. (a) I will get the job if my score is in the top 7%. let X denote the score of an applicant. We look for the smallest possible real number A so that $P(X \geq A) \leq 0.07$, in other words: above what limit value can we find the top 7% of all the scores.

Since the distribution is not known, try to use Chebyshev's inequality. Find an $\varepsilon > 0$ such that

$$P(m - \varepsilon < X < m + \varepsilon) = P(60 - \varepsilon < X < 60 + \varepsilon) \geq 1 - \frac{36}{\varepsilon^2} = 0.93$$

Indeed, in this case $P(X \geq 60 + \varepsilon) \leq 0.07$ is automatically fulfilled, thus $60 + \varepsilon$ is a good candidate for the limit value A. Based merely on Chebyshev's Theorem this ε is the lowest that we can guarantee. From the equation

$$1 - \frac{36}{\varepsilon^2} = 0.93$$

we get $\varepsilon^2 \approx 514.28$, that is $\varepsilon \approx 22.677$. The inequality $60 + 22.677 < 83$ implies that I will certainly get the job.

ATTENTION! If we had come up with $m + \varepsilon > 83$, it still would have been possible to get the job, because Chebyshev's Theorem gives a relatively rough estimate. Not to mention that we ignored the scores below $m - \varepsilon$. The fact is that in this case getting the job cannot be guaranteed (at most we may say it is likely if the difference is small).

(b) If the scores are approximately normally distributed with parameters $m = 60$ and $\sigma = 6$, then for the limit value A we get the inequality

$$P(X \geq A) = 1 - F(A) = 1 - \Phi\left(\frac{A - 60}{6}\right) \leq 0.07$$

This leads to the equation

$$\frac{A - 60}{6} = 1.48$$

from which $A \approx 68.88$. With the extra information we see that even a score of 69 points is enough to get the job.

(c) If X is uniformly distributed on an interval $[a, b]$, then for the unknown endpoints a and b

$$\frac{a + b}{2} = 60 \quad \text{és} \quad \frac{(b - a)^2}{12} = 36$$

From this system of equations we get $b - a \approx 20.6$, and the solutions are $a \approx 49.7$ and $b \approx 70.3$. The top 7% of this interval is $[68.8, 70.3]$. This means that a score of 69 is again definitely enough to get one of the open positions.

Problem 63

An insurance company has 10 000 clients. In a given year every client files a claim with probabilty $p = 0.002$ independently from each other. Find a lower bound and an upper bound so that with 90% certainty the number of incoming claims will be between the bounds in the given year.

Hint. Let X the number of incoming claims in the given year. This is clearly a Bernoulli experiment, therefore, X is binomially distributed with parameters $n = 10000$ and $p = 0.002$. Thus, $m = E(X) = np = 20$, and $Var(X) = \sigma^2 = np(1-p) = 20 \cdot 0.998 \approx 20$, and hence $\sigma \approx 4.48$.
By Chebyshev's Theorem for every $\varepsilon > 0$ we have

$$P(|X - m| < \varepsilon) \geq 1 - \frac{\sigma^2}{\varepsilon^2} = 0.9$$

The best (lowest) value of ε is obtained if

$$1 - \frac{20}{\varepsilon^2} = 0.9$$

that is $\varepsilon^2 = 200$, and $\varepsilon \approx 14.14$. Conclusion: with 90% certainty the number of claims will be between 5 and 35 in the given year.